WAVES / ACOUSTICS

Ripples on a lake, musical sound, seismic shakes triggered by an earthquake – all these are wave phenomena. Waves is one of the most important physical phenomena in technics.

Wave occur when a system is disturbed from equilibrium and when the disturbance can travel from one region of the system to another.

When the wave propagates it carries energy.

ex: – energy in the light waves from the sun warms the Earth
– energy of seismic waves cracks buildings

There are different kind of waves:

→ MECHANICAL WAVES need a MEDIUM to propagate.

→ ELECTROMAGNETIC WAVES can propagate without medium, even in vaccum
- light
- radio waves
- IR, UV radiation...
- X Rays...

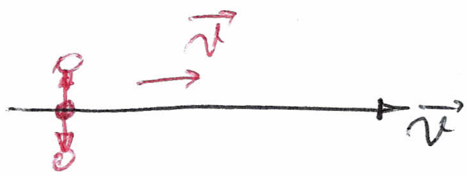
1 TYPES OF MECHANICAL WAVES

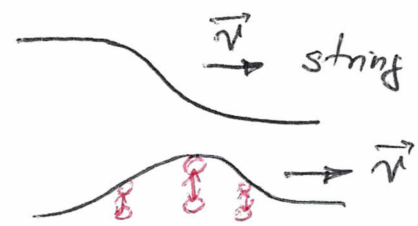
A mechanical wave is a disturbance that travels through some material or substance called medium of the wave.

As the wave travel to the medium, the particles that constitute the medium undergo displacements of various types, depending on the nature of the wave.

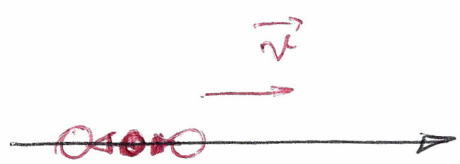
The figure below illustrate three varieties of mechanical waves:

→ <u>transverse waves</u> : displacement of the medium particles are transverse, perpendicular to the direction of the wave propagation.

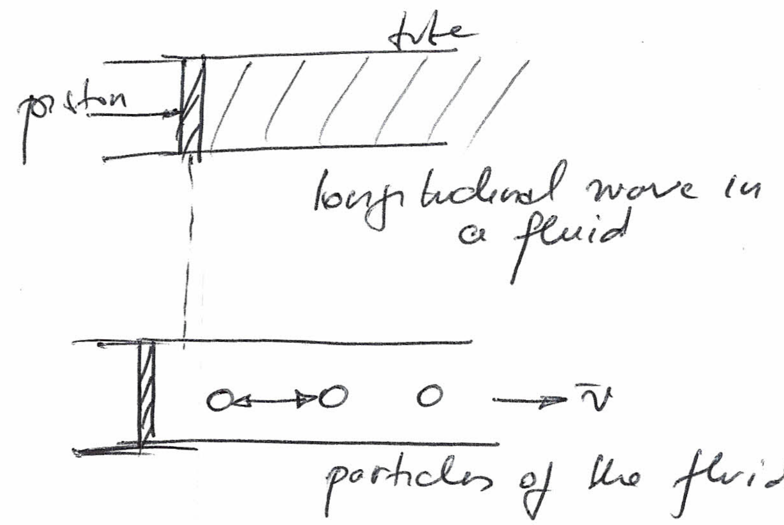




→ <u>longitudinal waves</u> : displacement of the medium are back and forth along the <u>same direction</u> that the wave travel



<u>ex</u> sound wave



→ There are propagating waves with both longitudinal and transverse components:

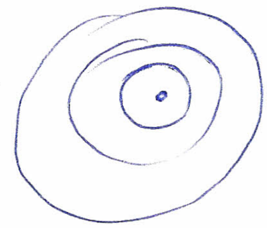
<u>ex</u> waves at the surface of a liquid

These three different types of waves have <u>common characteristics</u>

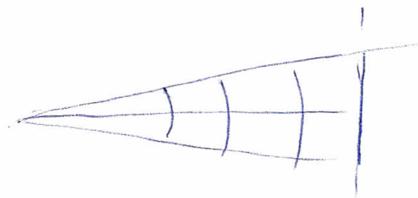
<u>WAVE FRONT</u> := the continuous line or surface including all the points in the space riched by the wave at the same instant through the medium

As a function of the shape of the wave front, the waves can be classified in

→ plane waves
→ circular waves
→ spherical waves



circular wave, perturbation initiated in the center.



at large distance to the center circular = plane.

plane wavefront

Wave characteristics

① Wave speed : – determined by the mechanical properties of the medium
– not the same with the speed of the medium's particles.

② The medium itself does not propagate, its individual particles undergo up-down or back-forth oscillations with respect to the wave propagation direction.

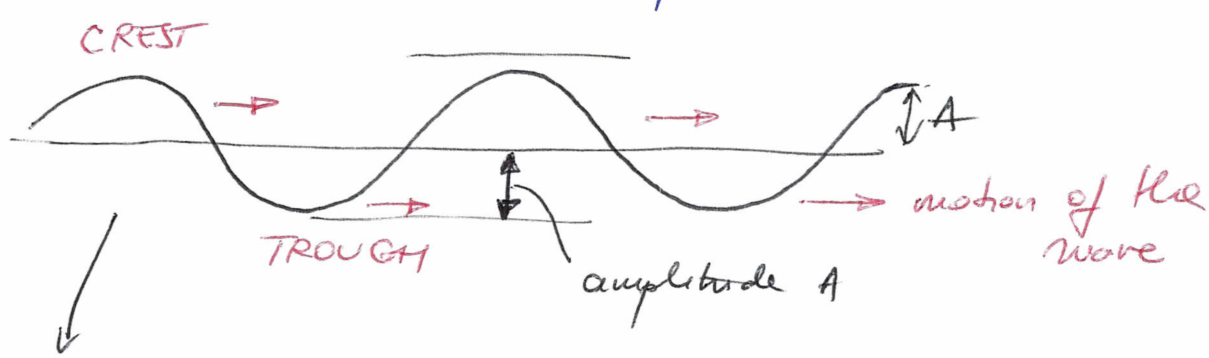
③ To set a wave we have to put energy or do work on the system. The wave motion transports the energy from one region of the system to another

⇒ wave transport energy but not matter from one region to another.

② PERIODIC WAVES

The transverse wave of a stroked string is an example of wave pulse, (when the string has been stroked just once). The tension in the stretched string restores the straight line shape after the pulse passed.

A more interesting situation develops when we give to the end of the string a repetitive periodic motion. Then, each particle of the string undergoes periodic motion as the wave propagates ⟹ periodic wave.



Periodic transverse wave: the end of the string is moved up ↔ down with SHM with:

→ amplitude A
→ frequency f
→ angular frequency ω
→ period $T = 1/f = 2\pi/\omega$

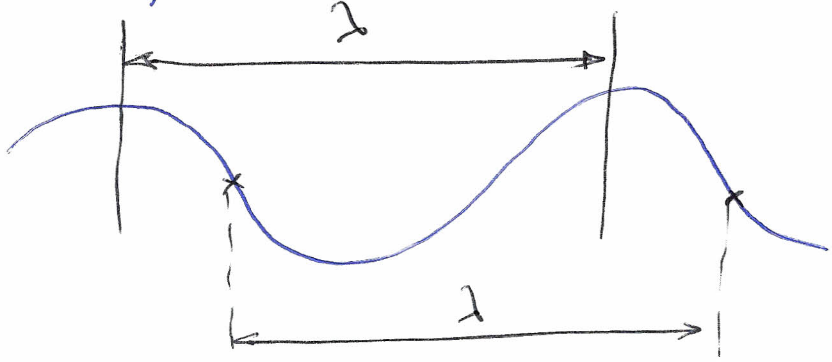
The wave that results has periodical, symmetrical crests and throughs (valleys).

Periodic waves with SHM are sinusoidal waves.

Any periodical wave can be represented as a combination of sinusoidal waves (see. Fourrier developement (series) in maths).

When a sinusoidal wave passes through a medium, any particle of the medium undergoes SHM.

For a periodic wave, the shape of the string has a repeated pattern



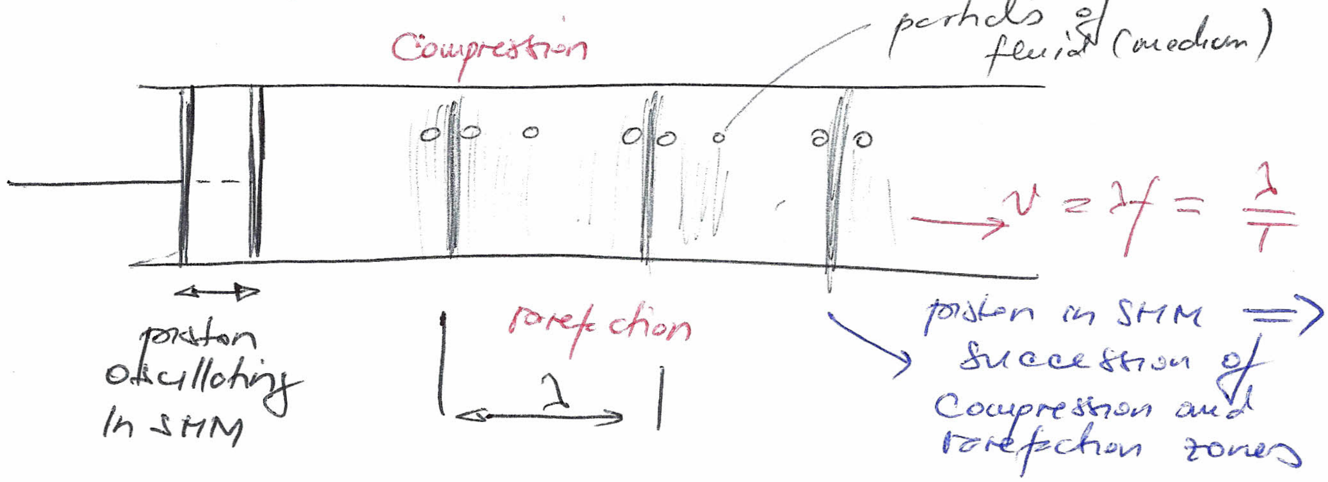
$\Rightarrow$ wavelength λ $\quad [\lambda] = m$

The wave-pattern travels with a constant speed v and advances a distance λ in a time T called period:

$$\boxed{v = \frac{\lambda}{T} = \lambda f}$$

Obs: waves on a string propagate 1D but all the concepts remain valid for 3D cases.

Periodic longitudinal waves



sound waves = longitudinal waves in air (fluid).

③ MATHEMATICAL DESCRIPTION OF A WAVE

Beyond the characteristics of a periodic wave described above (v, λ, f, ω, T) often we need a more detailed description of the position and motion of individual particle in the medium during wave propagation.

$\Longrightarrow$ wave function

ex For a transverse wave $y = y(x,t)$ describes the displacement in y of points along X axis at time t.

Wave function for a sinusoidal wave

$$\boxed{\begin{aligned} y(x) &= y(x+\lambda) \\ y(x,t) &= y(x, t+T) \end{aligned}}$$

general characteristics for any type of periodical wave

$$\boxed{y(x,t) = A\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]}$$

sinusoidal wave moving along $+x$ direction

If we introduce:

$$\boxed{k = \frac{2\pi}{\lambda}}$$

the wave number

$[k] = \frac{rad}{m}$

and $T = \frac{2\pi}{\omega}$

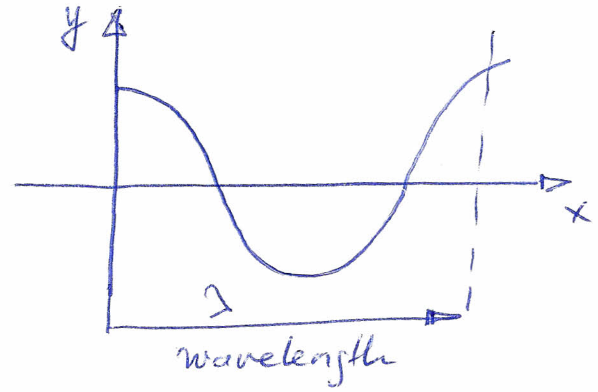
$$\Rightarrow \boxed{y(x,t) = A\sin(kx - \omega t)}$$

wave propagating along $+X$ direction

$$\boxed{y(x,t) = A\sin(kx + \omega t)}$$

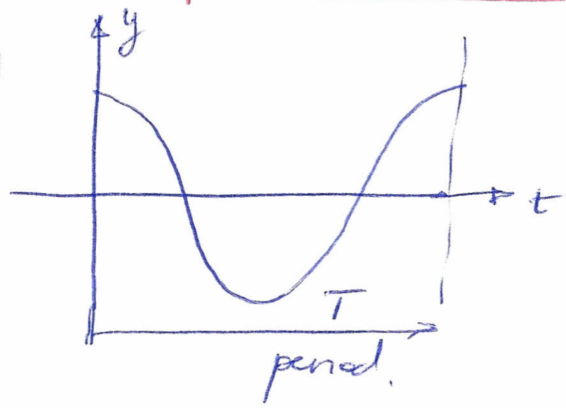
wave propagating along $-x$ direction

Graphing the wave function $\boxed{y = A\cos(kx \pm \omega t)}$



at $t=0$ the curve represents the shape of the string

$$\boxed{y(x, t=0) = A\cos 2\pi \frac{x}{\lambda}}$$



time evolution of particle position y at $x=0$.

$$\boxed{y(x=0, t) = A\cos 2\pi \frac{t}{T}}$$

The quantity $(kx \pm \omega t)$ is called PHASE

It plays the role of an angular quantity (measured in radians).

For a crest: $kx \pm \omega t = 0, 2\pi, 4\pi, \ldots, = 2n\pi$

$$\cos(kx \pm \omega t) = 1 \Rightarrow y = +A$$

For a valley: $kx \pm \omega t = \pi, 3\pi, 5\pi, \ldots (2n+1)\pi$

$$\cos(kx \pm \omega t) = -1 \Rightarrow y = -A.$$

The wave speed → the speed to move along with the wave to keep alongside a point with a given phase (i.e. a particular crest) ⟺ keep the phase constant

$$\Rightarrow kx - \omega t = \text{const} \Leftrightarrow \frac{d}{dt}(kx - \omega t) = 0 \Leftrightarrow$$

$$\boxed{\frac{dx}{dt} = \frac{\omega}{k} = v}$$

→ speed of wave or phase speed.

$$\boxed{v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f}$$

Diferential equation of wave

→ particle velocity and acceleration in sinusoidal wave

From: $y = A\cos(kx - \omega t)$ we can deduce the particle velocity at any instant t

Particle velocity

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A \sin(kx - \omega t)$$

- periodical function ⇒ SHM
- maximum value $(v_y)_{max} = \pm \omega A$
- may be larger, equal or smaller than the wave speed v, depending on A and ω.

Particle acceleration

$$a_y(x,t) = \frac{\partial v_y(x,t)}{\partial t} = \frac{\partial^2 y(x,t)}{\partial t^2}$$

$$\boxed{a_y(x,t) = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x,t)}$$

equivalent with what we got for SHM

We can also calculate derivatives for $y(x,t)$ as a fct of x

$\frac{\partial y(x,t)}{\partial x}$ = slope of string at position x at time t

$\frac{\partial^2 y(x,t)}{\partial x^2}$ = curvature of the string.

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -k^2 A \cos(kx - \omega t) = -k^2 y(x,t)$$

From: $\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 y(x,t) \\ \dfrac{\partial^2 y(x,t)}{\partial x^2} = -k^2 y(x,t) \end{cases}$; $v = \dfrac{\omega}{k}$

$$\Rightarrow \boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}}$$ wave equation

→ one of the most general equations in physics
→ valid in the most general situation, whether the wave is periodical or not
→ electric and magnetic field satisfy wave equation with $v = c$ (speed of light) – light is an electromagnetic wave

Generalizing propagation in a 3D medium

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} + \frac{\partial^2 y}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

$$\boxed{\nabla^2 y(x,t) - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0}$$ general differential eq. of waves.

4 SPEED OF A TRANSVERSE WAVE

One of the most important wave properties is the wave speed. Light waves have a much greater propagation speed ($3 \cdot 10^8$ m/s) than sound in the air (344 m/s); that's why we see first a lightening and hear the thunder later...

We would like to correlate the speed of a wave in a medium and some characteristics properties of the medium.

The speed of mechanical waves can be demonstrated to be given by the following general equation:

$$v = \sqrt{\frac{\text{Restoring force returning the particle to equilibrium}}{\text{Inertia resisting the return to equilibrium}}}$$

Transverse waves in a string

→ the restoring force is the tension in the string F, it tends to bring back the string in the unperturbed position

→ the inertia resisting return to equilibrium is the mass of the string, more precisely, the mass/unit length

$\mu = dm/dx$ or $\mu = \frac{m}{\ell}$ — string length.

$$\Rightarrow \boxed{v = \sqrt{\frac{F}{\mu}}}$$

Longitudinal sound waves

The gass pressure provides the force to bring back the system in the unperturbated state after the wave passed through. The inertia is provided by the gas density

$$\Rightarrow \boxed{v = \sqrt{\frac{B}{\rho}}} = \sqrt{\frac{\gamma p_0}{\rho}}$$

(acoustics) → see next chapter for details

B = bulk modulus of the medium

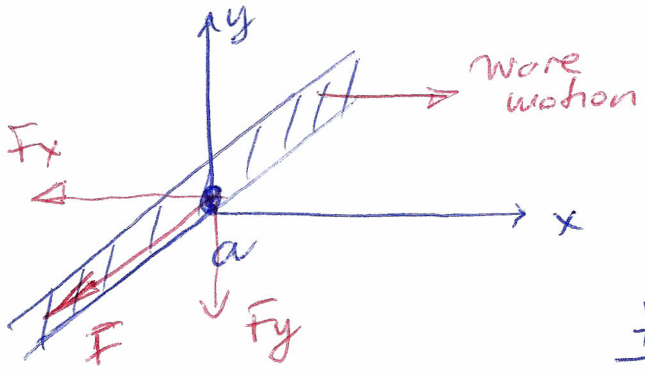
p_0 – equilibrium pressure of the gas

γ = ratio of heat capacities

5 Energy in wave motion

Wave carry energy in their motion.

We focus on a transverse wave on a string:



point on a string carrying a

$$\frac{F_y}{F} = -\frac{\partial y(x,t)}{\partial x}$$

the power: $P = F_y v_y$ (transverse force × transverse velocity)

$$P(x,t) = -F \frac{\partial y(x,t)}{\partial x} \cdot \frac{\partial y(x,t)}{\partial t}$$

for a sinusoidal wave: $y = A\cos(kx - \omega t)$

$$\Rightarrow \quad P(x,t) = F k \omega A^2 \sin^2(kx - \omega t)$$

$$v = \frac{\omega}{k} \; ; \quad v^2 = \frac{F}{\mu}$$

$$\Rightarrow \boxed{P(x,t) = \sqrt{F\mu}\, \omega^2 A^2 \sin^2(kx - \omega t)}$$

maximum power: $\sin^2 \omega t = 1 \Rightarrow \boxed{P_{max} = \sqrt{F\mu}\, \omega^2 A^2}$

average power: $\langle \sin^2 \omega t \rangle = 1/2 \Rightarrow \boxed{P_{max} = \frac{1}{2}\sqrt{F\mu}\, \omega^2 A^2}$

The average rate of energy transfer is

$\sim \omega^2 A^2$ valid for any mechanical wave

For electromagnetic waves $P \sim A^2$ but independent on ω.

Wave intensity

Wave on a string carry energy in 1D of space. But other types of mechanical waves (i.e. seismic waves, sound waves in air) carry energy in all 3D directions of space.

For 3D waves, we define the intensity: I to be the time average at which energy is transported by the wave per unit area across a surface perpendicular to the propagation direction of the wave

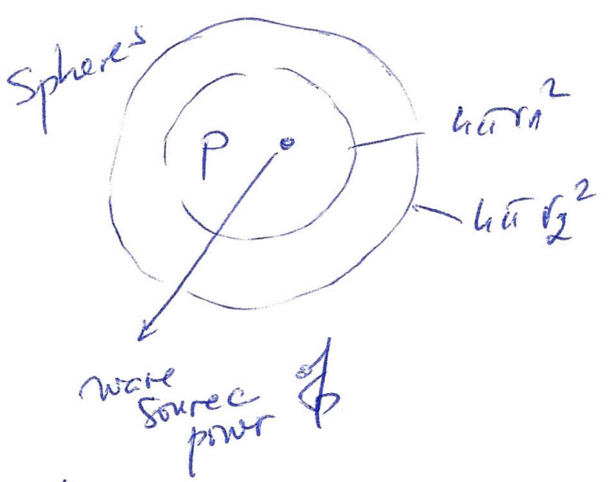
$$\boxed{I = \frac{\text{average power}}{\text{unit area}}} \qquad [I] = \frac{W}{m^2}$$

If waves spread out equally in all directions from a source, the intensity:

$$\boxed{I \sim 1/r^2}$$ (comes from energy conservation)

$$I_1 = \frac{P}{4\pi r_1^2};\ I_2 = \frac{P}{4\pi r_2^2} \Rightarrow I_1 \cdot 4\pi r_1^2 = I_2\, 4\pi r_2^2$$

~~if no power lo~~



$$\Rightarrow \boxed{\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}}$$

<u>Obs</u>: In case of 1D propagation, atenuation is less than $1/r^2$ (see next chapter of acoustics).

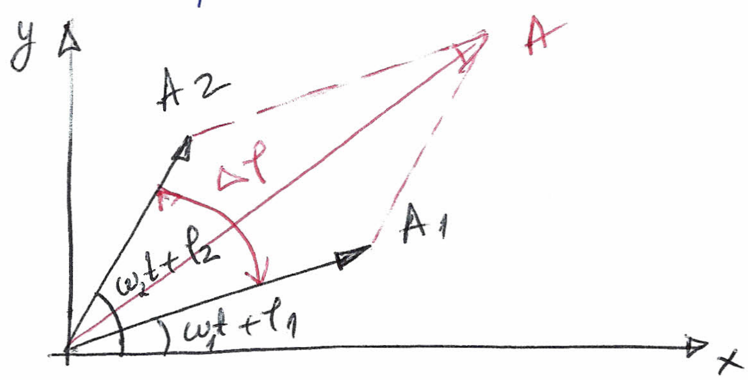
INTERFERENCE of oscillations / waves

$$y_1 = A_1 \sin(\omega_1 t + \varphi_1)$$

$$y_2 = A_2 \sin(\omega_2 t + \varphi_2)$$

$$\boxed{y = y_1 + y_2 = A \sin(\omega t + \varphi)}$$

Phasor representation



$$\Delta\varphi = (\omega_2 - \omega_1)t + (\varphi_2 - \varphi_1)$$

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos\Delta\varphi$$

Value of average $\langle A^2 \rangle \sim$ wave intensity I

$$\langle A^2 \rangle = A_1^2 + A_2^2 + 2A_1A_2 \frac{1}{T}\int_0^T \cos\left[(\omega_2 - \omega_1)t + (\varphi_2 - \varphi_1)\right] dt$$

the integral is $= 0$ for $\omega_1 \neq \omega_2$

in this case, no interference and

$A^2 = A_1^2 + A_2^2 \iff \underline{I = I_1 + I_2}$

If $\omega_1 = \omega_2$, in order to have interference one must have: $\boxed{\varphi_1 - \varphi_2 = \text{constant in time}}$

$\Rightarrow$ coherent waves (phase difference const. in time)

Then $\langle A^2 \rangle = A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1)$

- $\boxed{\Delta\varphi = \varphi_2 - \varphi_1 = 2n\pi}$ $\Rightarrow \cos(\varphi_2 - \varphi_1) = 1$

  $n = 0, 1, 2, \ldots$ maximum of interference

  $I \sim A^2 = max = (A_1 + A_2)^2$

- $\boxed{\Delta\varphi = \varphi_2 - \varphi_1 = (2n+1)\pi}$ $\Rightarrow \cos(\varphi_2 - \varphi_1) = -1$

  $n = 0, 1, \ldots$

  $I \sim A^2 = min = (A_1 - A_2)^2$

In terms of λ ; $\varphi = \omega t - Kx$

$\Delta\varphi = 2n\pi \Rightarrow \Delta\varphi = K\Delta x = 2n\pi$

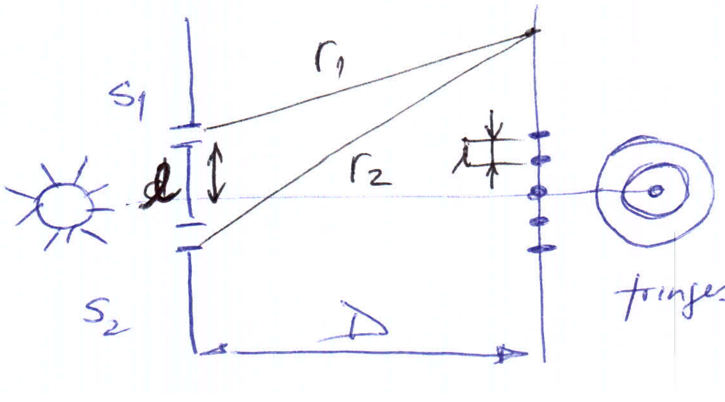
$= \frac{\cancel{2\pi}}{\lambda}\Delta x = \cancel{2}n\cancel{\pi}$

$\Rightarrow \boxed{\Delta x = 2n\frac{\lambda}{2}}$ maximum of interference

analogously $\boxed{\Delta x = (2n+1)\frac{\lambda}{2}}$ minimum of interference

Obs Interference is a phenomenon common to waves, independent on their nature

Experiment of Thomas Young



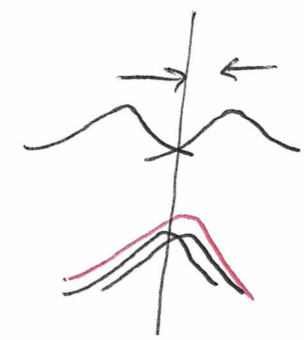
max. and min fringes on screen

$i = d_{n+1} - d_n$

$= \frac{\lambda D}{e}$

⑥ WAVE INTERFERENCE, BOUNDARY CONDITIONS

The interference represents the overlapping between waves. The interference happens when several waves pass through the same area, at the same time.



Principle of superposition

$$\boxed{y(x,t) = y_1(x,t) + y_2(x,t)}$$

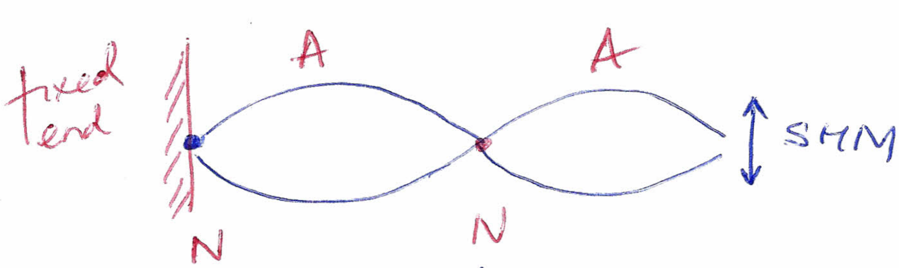
resulting oscillation is the vector sum of individual oscillations.

When a wave reaches the boundary of a medium it is REFLECTED. (i.e. echo, when yelling away from a cliff or building) $\Rightarrow$ bondary conditions.

⑦ STANDING WAVES IN A STRING

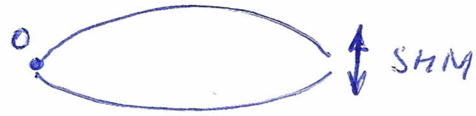
→ string fixed at left side

⇒ right side in SHM

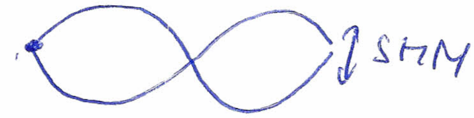


$\Rightarrow$ standing wave pattern with nodes ($y=0$) and antinodes ($y=max$) when 2 waves travelling in opposite direction interfere

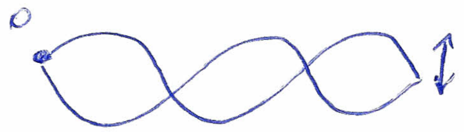
Time exposure of vibrating string



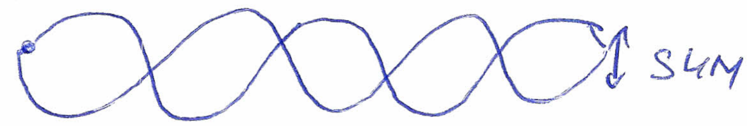
string = $\frac{\lambda}{2}$ length



string = λ length



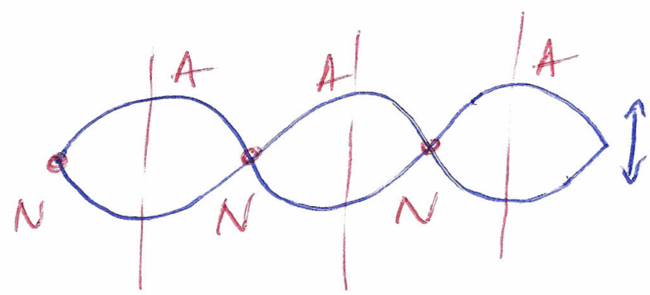
string = $\frac{3\lambda}{2}$ length



string = 2λ length

There are particular points called:

- NODES (N) that never move at all
- ANTINODES (A) : maxima of movement



Because the wave pattern doesn't not appear to move in either direction along the string, it is called STANDING WAVE.

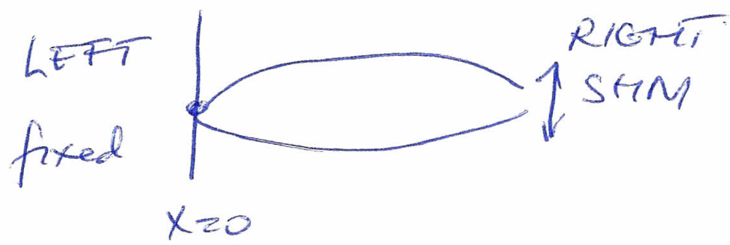
(≠ travelling wave)

The principle of superposition explains how the incident and the reflected wave at fixed end combine to form a standing wave:

- at nodes N → destructive interference
- at antinodes A → constructive interference

The wave reflected from a fixed end reverts the sign, so we give a negative sign to one of the waves:

$$\Rightarrow y_1(x,t) = -A\cos(kx+\omega t)$$
$$y_2(x,t) = A\cos(kx-\omega t)$$



$$y(x,t) = y_1(x,t) + y_2(x,t) = A[-\cos(kx+\omega t) + \cos(kx-\omega t)]$$

$$\cos(a-b) = \cos a\cos b - \sin a \sin b$$

$$\Rightarrow \boxed{y(x,t) = 2A\sin kx \sin\omega t} \quad \Leftrightarrow$$

$$\boxed{y(x,t) = (A_{SW}\sin kx)\sin\omega t}$$

standing wave of a string fixed at $x=0$

$\rightarrow$ standing wave amplitude

$A_{SW} = 2A$ = twice the amplitude of original travelling waves

$$y(x,t) = \boxed{\text{function of } x} \times \boxed{\text{function of time}}$$

$A_{SW}\sin kx$

at each instant of time the shape of the string is a sine shape

but unlike a travelling wave, the standing wave stays in the same position oscillating up and down as described by $\sin\omega t$ factor.

Each point of the string undergoes SHM, all the points between two pairs of nodes oscillate in phase.

→ The position of nodes:

$\sin kx = 0 \Rightarrow kx = n\pi \Rightarrow x = \frac{n\pi}{k}$

but $k = 2\pi/\lambda$ $\Rightarrow$

$$\boxed{x = n\frac{\lambda}{2}}$$

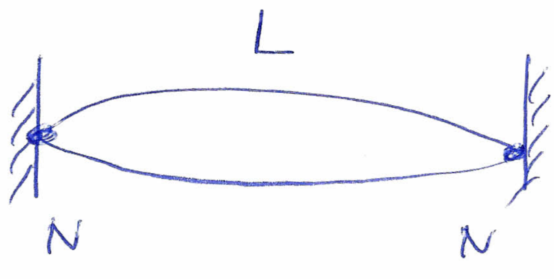
position of nodes in a string fixed at $x = 0$

Obs The standing wave, unlike a travelling wave does not carry energy from one end to the other. The two waves that form the standing wave carry equal amount of power in opposite directions ⇒ total local average power is zero.

(8) Normal modes of a string

→ String of length L rigidly fixed at both ends (ex. musical instruments: guitar, violin...)

When a guitar string is plucked ⇒ wave produced in the string ⇒ standing wave on the string ⇒ sound wave in the air...



both end fixed ⇒ boundary conditions = Nodes at both ends.

The condition for nodes:

$$\boxed{L = n\frac{\lambda}{2}}$$

$$\Rightarrow \boxed{\lambda_n = \frac{2L}{n}}$$ $n = 1, 2, \ldots$ possible values for λ in the string L

Obs: waves may exist for any λ but not standing waves

corresponding to that λ_n $\Rightarrow \boxed{f_n = \frac{v}{\lambda_n}}$ frequencies:

The smallest f_1 corresponds to longer λ_n

$$\Rightarrow \boxed{f_1 = \frac{v}{2L}}$$ $\Rightarrow$ fundamental frequency

The other frequencies:

$$\boxed{f_n = n\frac{v}{2L}}$$ are called harmonics

$n = 2, 3, \ldots$

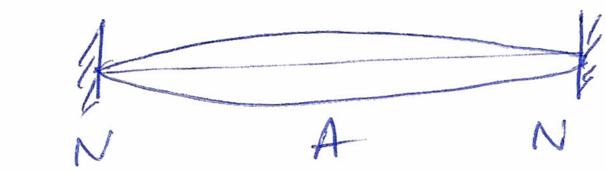
For a string fixed at $x = 0$ and $x = L$ the wave function $y(x,t)$ of the standing wave is:

$$\boxed{y_n(x,t) = A_{SW} \sin k_n x \sin \omega_n t}$$

with $\omega_n = 2\pi f_n$; $k_n = \frac{2\pi}{\lambda_n}$

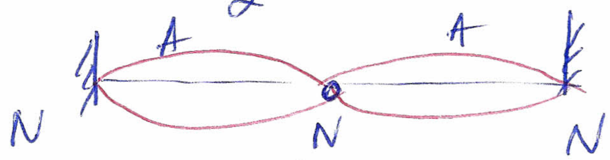
A normal mode of an oscillating system is a motion in which all the particles of the medium moves sinusoidally with same frequency.

The first four normal modes of a string:



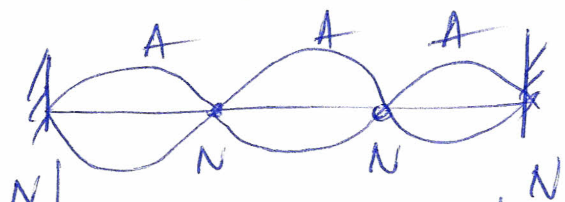
$n=1$ fundamental frequency f_1

$$\frac{\lambda}{2} = L$$



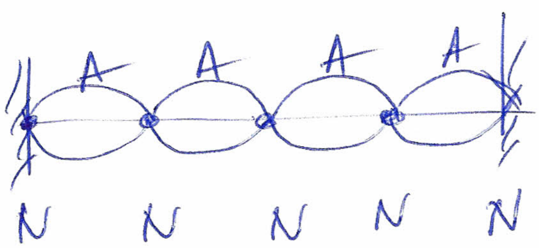
$n=2$ second harmonic f_2

$$\frac{2\lambda}{2} = L$$



$n=3$ third harmonic f_3

$$3\frac{\lambda}{2} = L$$



$n=4$ fourth harmonic f_4

$$4\frac{\lambda}{2} = L$$

String instruments

$$f_1 = \frac{v}{2L}$$ but $v = \sqrt{\frac{F}{\mu}}$ $\Rightarrow$

$$\boxed{f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}}$$

frequency depend on properties of string

- reducing L increase f (pressing with finger string of guitar, violin...)
- increasing tension F increasing frequency.

- increasing the mass / unit length (μ) decreases frequency $\Rightarrow$ to produce bass, thicker string.

Complex standing waves

If we could perfectly pluck a string in one of its normal modes, it would vibrate only with the frequency of that mode. But this is an ideal case, in reality, modes are mixed: $\Rightarrow$ HARMONIC CONTENT

→ fundamental and many harmonics are present in the same vibration

→ motion is a superposition of many normal modes

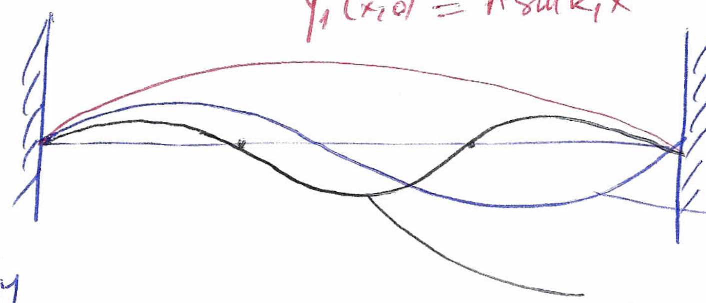
→ the sound in the air, as well as the standing wave in the string have similar harmonic content, which depend on how the string has been initially set in motion

It is possible to represent any possible complex motion of the string as a superposition of normal ~~and~~ modes motion. Finding this representation for a given vibration is called harmonic analysis.

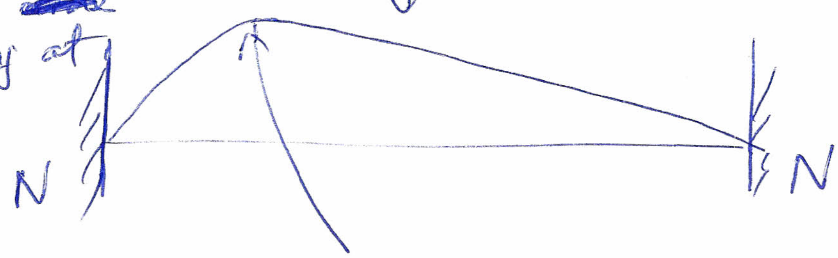
↳ Fourier series decomposition

$$\boxed{y(x,t) = \sum_{j=1}^{n} \left[A_j \cos(j\,kx) + B_j \sin(j\,kx)\right]}$$

Standing wave produced by plucking a guitar string at $x = \frac{L}{4}$



$\Downarrow$



$$y(x,0) \simeq y_1(x,0) + y_2(x,0) + y_3(x,0)$$

Obs: Including additional sinusoidal functions further improves the representation