PERIODIC MOTION
OSCILLATIONS

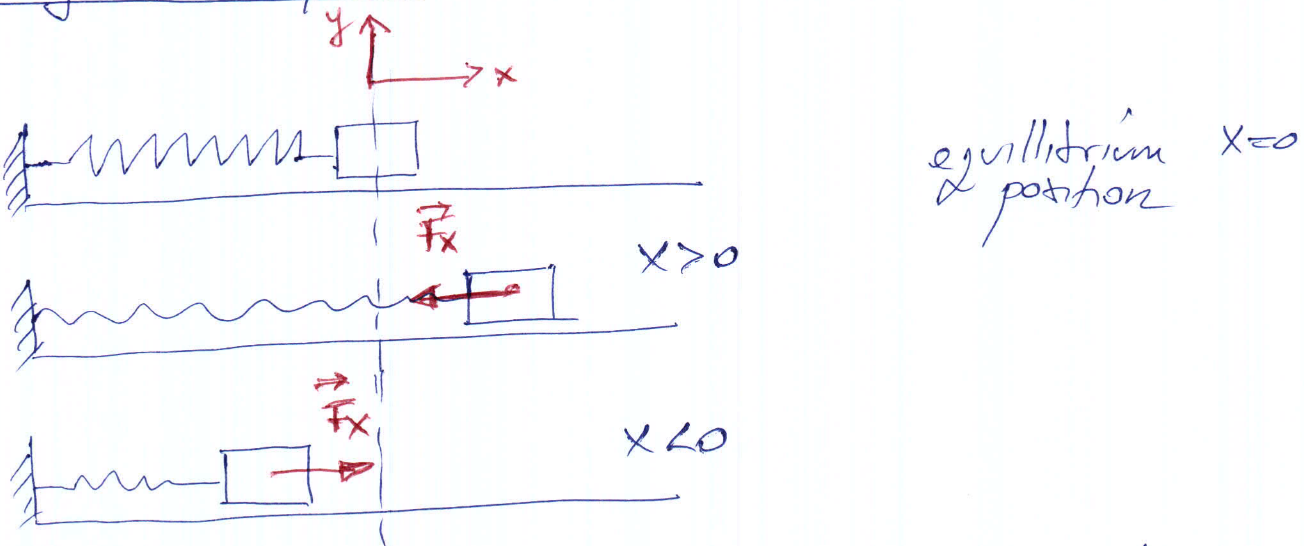
There are motions repeating themselves over and over: vibrations of a quartz crystal in a watch, swinging pendulum in a clock, back and forth motion of pistons in an engine... This kind of motion is a periodic motion or oscillation. Understanding of oscillations is essential for later studies on waves, sound, alternating electric current, light and electromagnetic field theory...

A body undergoing periodical motion always have a STABLE EQUILLIBRIUM POSITION. When it leaves the equillibrium position a force or a torque appears which tends to restore the equillibrium configuration

$\Rightarrow$ RESTORING FORCE.

1 Describing oscillations

spring-mass system:



→ when the body is displaced from equillibrium at x=0 the spring exerts a restoring force backwards to the equillibrium position

$\Rightarrow$ oscillation = periodic motion around equillibrium

Amplitude. Period. Frequency. Angular frequency.

- **Amplitude** A is the maximum displacement from equilibrium

$$A = \max |x| \qquad [A]_{SI} = m$$

Complete vibration or 1 cycle is one complete round trip from

$$O \to A \to O \to -A \to O$$

obs motion from $-A$ to A is $1/2$ cycle.

- **Period** T is the time for one cycle, it is allways positive

$$[T]_{SI} = 1\,s$$

- **Frequency** f, ν is the numer of cycles in the unit of time

$$\boxed{f = \frac{1}{T}} \qquad [f]_{SI} = 1/s = 1\,Hz.$$

named after the German scientist Heinrich Hertz, pioneer in investigating electromagnetic waves

- **Angular frequency**

$$\boxed{\omega = 2\pi f = \frac{2\pi}{T}} \qquad [\omega]_{SI} = rad/s$$

Simple harmonic motion (SHM)

- occurs when the restoring force is directly proportional to the displacement from equilibrium

$$\boxed{F_x = -kx}$$

Hook's law for ideal spring

The acceleration: $F_x = -kx = ma_x \Rightarrow a_x = -\frac{k}{m}x$

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

− sign means that acceleration and displacement have opposite sign.

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

diff. eq. of the harmonic oscillator

2nd order differential equation whose solution will be $x = x(t)$

$$\frac{k}{m} = \omega^2$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

Frequency: $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Period: $T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

- period and frequency in SHM are completely determined by the mass m and the elastic constant k
- T and f does not depend on the amplitude A

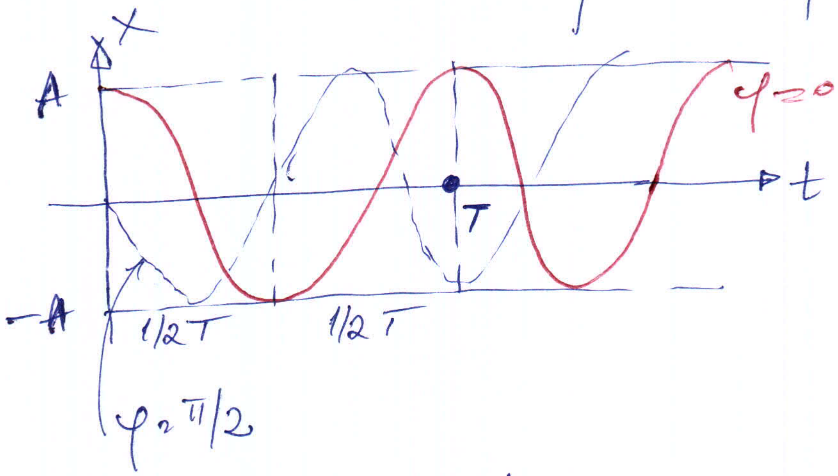
• Displacement, velocity and acceleration

The solution of the differential equation is:

$$\boxed{x(t) = A\cos(\omega t + \varphi)}$$

→ phase angle

periodic function of time.



Velocity: $v_x = \dfrac{dx}{dt} = -\omega A \sin(\omega t + \varphi)$

oscillates between $\begin{cases} v_{max} = +\omega A \\ v_{min} = -\omega A \end{cases}$

Acceleration: $a_x = \dfrac{dv_x}{dt} = \dfrac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \varphi) = -\omega^2 x$

oscillates between $+\omega^2 A$, $-\omega^2 A$

obs: $x(t) = 0$ $\quad v = v_{max}$; $a_x = 0$

$x(t) = \pm A$ $\quad v = 0$; $a_x = a_{max}$.

Energy in SHM

kinetic energy: $K = \frac{1}{2} m v_x^2$

potential energy: $U = \frac{1}{2} k x^2$

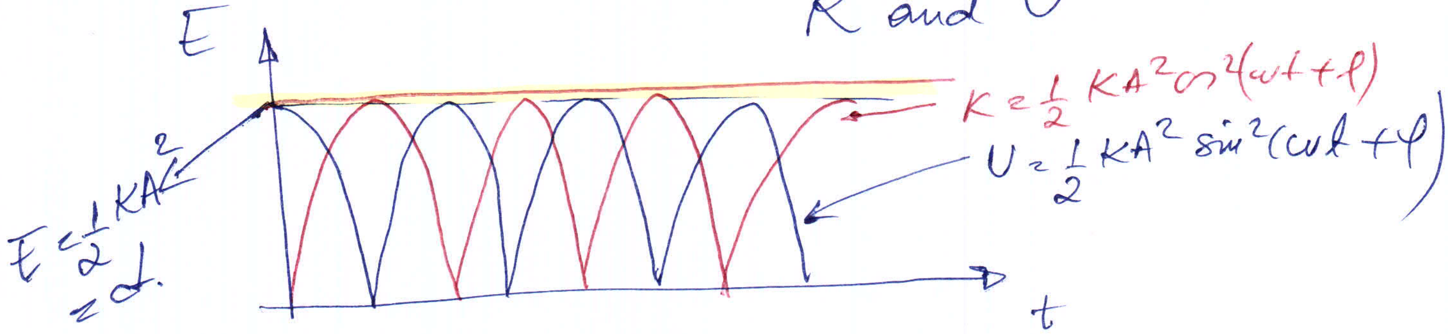
$$\begin{cases} x = A\cos(\omega t + \varphi) \\ v_x = -\omega A \sin(\omega t + \varphi) \\ \omega^2 = \frac{k}{m} \end{cases}$$

$$\begin{cases} K = \frac{1}{2} m^2 \omega^2 A^2 \sin^2(\omega t + \varphi) = \frac{1}{2} k A^2 \sin^2(\omega t + \varphi) \\ U = \frac{1}{2} k A^2 \cos^2(\omega t + \varphi) \end{cases}$$

$$\boxed{E = K + U = \frac{1}{2} k A^2}$$

The total energy in SHM is conserved.

↳ continuous conversion between K and U



if $A = ct$ in time (SHM) $\Rightarrow E = ct.$

DAMPED OSCILLATIONS

The idealized oscillating systems discussed above are frictionless. All the implicated forces are conservative, and so, the total mechanical energy is constant, and the system once set in motion will oscillate forever to oscillate with $A = ct.$

Real-world systems allways have some dissipative forces ⇒ oscillations die unless we provide external energy to compensate dissipation

i.e. mechanical pendullum of a clock continues to oscillate because potential energy stored in the spring or a hanging weight system replaces the mechanical energy lost due to friction between the pivot and gears.

The decrease of the amplitude caused by dissipative forces is called damping, and the corresponding motion is called damped oscillation.

The simplest case to analyse in detail is a simple harmonic oscillator with a frictional damping force that is simply proportional to the velocity of the oscillating body:

$$F_x = -bv_x \quad ; \quad b = \text{constant describing the strength of the damping force:}$$

⇒ net force:

$$\sum F_x = -kx - bv_x = -kx - b\frac{dx}{dt}$$

Newton 2nd law:

$$-kx - bv_x = ma_x$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \iff$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x + \frac{b}{m}\frac{dx}{dt} = 0 \quad ; \quad \omega_0^2 = \frac{k}{m}$$

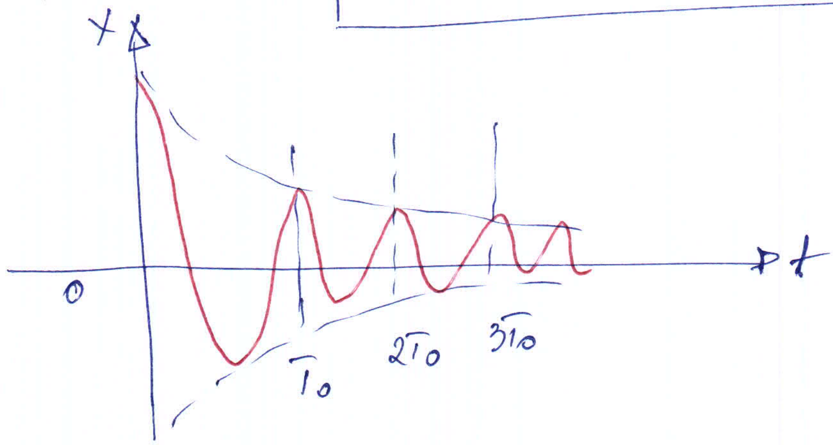
$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega_0^2 x + \frac{b}{m}\frac{dx}{dt} = 0}$$

differential equation similar with the one of SHM but with the aditionall linear term $+\frac{b}{m}\frac{dx}{dt}$

If the damping is relativelly small, the solution is:

$$\boxed{x = Ae^{-\frac{b}{2m}t}\cos(\omega' t + \varphi)}$$

$$\boxed{\omega' = \sqrt{\frac{K}{m} - \frac{b^2}{4m^2}}}$$



Obs: (1) $\omega' = \sqrt{\frac{K}{m} - \frac{b^2}{4m^2}} < \omega_0$

$$A = A_0 e^{-\frac{b}{2m}t}$$

not const. but decreasing in time

2) $\omega' = 0$ when $\frac{K}{m} = \frac{b^2}{4m^2}$ $\Rightarrow$

$$\boxed{b = 2\sqrt{Km}}$$

this condition is called critical damping

The system no longer oscillates but returns to equilibrium directly

If $b > 2\sqrt{Km}$ $\Rightarrow$ overdamping
no oscillation but the system returns to equilibrium more slowly than in critical damping

$b < 2\sqrt{Km}$ $\Rightarrow$ underdamping
the system oscillates with steadily decreasing amplitude

Energy in damped oscillations

$$E = \frac{1}{2} m v_x^2 + \frac{1}{2} K x^2$$ – varies in time

$$\frac{dE}{dt} = m v_x \cdot \frac{dv_x}{dt} + \frac{1}{\cancel{2}} \cancel{2} K x \frac{dx}{dt}$$

$$= v_x (m a_x + K x) = v_x (\cancel{-Kx} - b v_x + \cancel{Kx})$$

$$= -b v_x^2 = (-b v_x) v_x = \underbrace{F_{damping} \, v_x}_{\text{damping power}}$$

$$\boxed{\frac{dE}{dt} = P_{damping}}$$

Important physical quantities for damped oscillations.

Damping logarithmic decrement

$$\theta = \ln \frac{A(t)}{A(t+T)} = \ln e^{\frac{b}{2m}T} = \frac{b}{2m} T$$

$$\boxed{\frac{b}{2m} = \delta}$$ damping parameter

Relaxation time

: the time after which the amplitude decays e times ($e = 2,71828$)

$$A(t+\tau) = \frac{A(t)}{e}$$

$$\Rightarrow \boxed{\tau = \frac{1}{\delta}}$$

measure of the life time of a damped oscillation

Quality factor Q

defines the speed of energy loss due to dissipation

$$Q = 2\pi \frac{E(t)}{E(t+T) - E(t)} = \frac{2\pi E}{\Delta E}$$

if introducing the expression for $E(t) \Rightarrow$

$$\boxed{Q = \frac{2\pi}{T}\tau = \omega_0 \tau}$$

if $\delta > \omega_0 \Rightarrow Q < 1$ $\Rightarrow$ unharmonical oscillations

$\delta < \omega_0 \Rightarrow Q \gg 1$ $\Rightarrow$ harmonical oscillations

Obs: The quantities f, τ, Q are characteristic for any damped oscillatory phenomena, independently of its nature: mechanical, electrical, electromagnetical

ex: electric circuits with inductance L, capacitance C, resistance R. There is a natural frequency of oscillation and the resistance R plays the role of the damping constant b.

FORCED OSCILLATIONS AND RESONANCE

A damped oscillator left to itself will stop moving after a while, all the energy being lost by dissipation. But, we can mentain the system in oscillation if applying a force that varies in time in a periodic cycling way with definite period and frequency (i.e. playground swing, pushing once each period)
$\Rightarrow$ keep swinging with constant amplitude.

- This additional force is called DRIVING FORCE.
- If we apply a periodically varying force with the frequency ω_d to a damped oscillator

$$\boxed{F(t) = F_{max} \cos \omega_d t}$$

the oscillation resulted is called forced or driven oscillation.

$\neq$ from the case of system alone which oscillates with $\omega' = \sqrt{\frac{K}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \delta^2}$

- In forced oscillation the system oscillates with the <u>driving angular frequency ω_d</u>. This may be not equal with the frequency ω' the system would have without driving force acting on it.

- The amplitude of forced oscillation is a function of the driving force angular frequency ω_d and reaches a maximum (peak) when ω_d is close to the natural frequency of the system
  $\Rightarrow$ <u>RESONANCE</u>

- We can deduce mathematically:

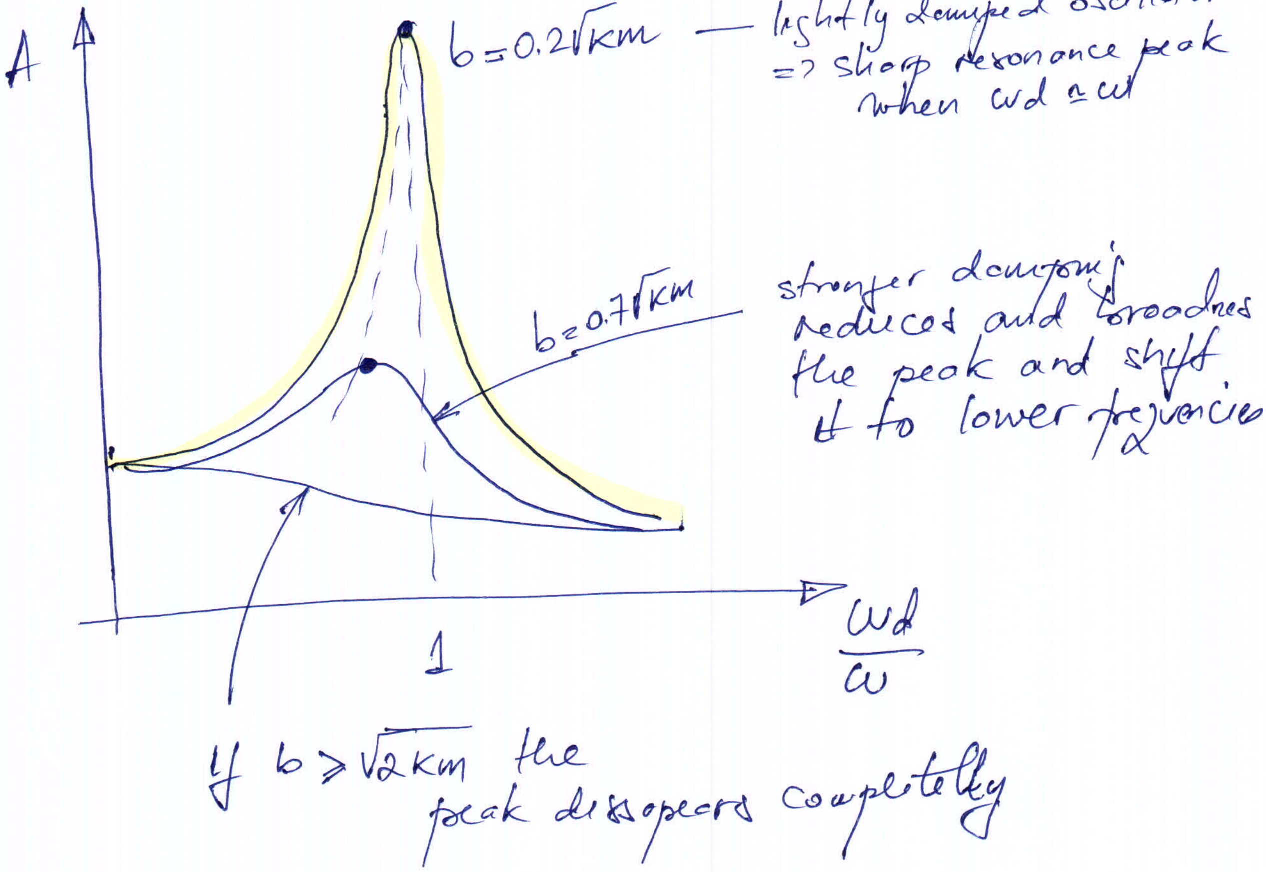
$$\boxed{A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}}}$$

when $k - m\omega_d^2 = 0$ the first term under the radical is zero. So A has a maximum near $\omega_d = \sqrt{\frac{k}{m}}$

The height of the curve at this point is proportional with $1/b$; the less damping, the higher the peak $\left(Q = \frac{\omega_0}{\delta} \right) \Rightarrow$ less damping = higher Q (quality factor)

At low frequency regime ($\omega_d = 0$) we get $A = \frac{F_{max}}{k}$. This corresponds to a constant force F_{max} and a constant displacement $A = \frac{F_{max}}{k}$ as we might expect from equilibrium

Graph of the amplitude of the forced oscillation as a function of ω_d. The horizontal axys shows the ratio $\frac{\omega_d}{\omega_0}$ where $\omega_0 = \sqrt{\frac{m}{k}}$ is the natural freqvency of undamped oscillator. Each curve has a different value of the damping constant b.



Resonance and its consequencies

The amplitude of forced oscillation <u>reaches a peak</u> when the f of the driving force is close to the natural freqvency of the system, ~~at~~ resonance. When the damping is small, the quality factor is large, and for $\delta \to 0$; $Q \to \infty$ $A \to \infty$.

This enhancement of amplitude at resonance may have <u>distructive consequences</u>.

→ a company of soldiers once destroyed a bridge by marching across in step; the frequency of their steps was close to the natural vibration frequency of the bridge, and the resulting oscillation was large enough to tear the bridge apart.

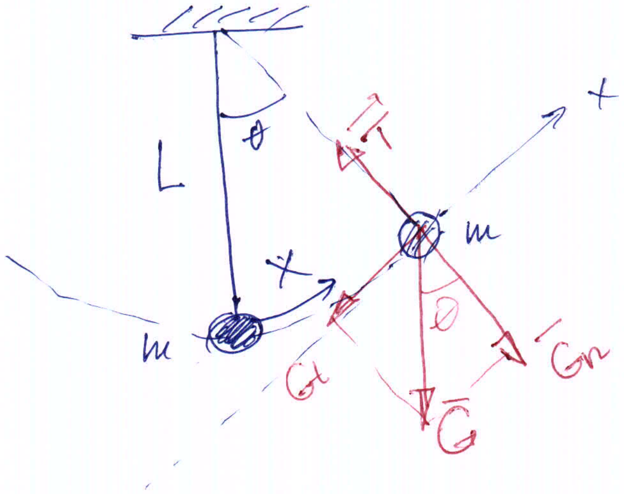
→ vibrations of engine of a particular airplane coincide in frequency with natural frequency of wings, ⇒ wings have broken.

⇒ necessity to avoid resonances in systems where amplification of amplitude can have negative consequences.

In some other applications, the maximum transfer of energy is riched at resonance, and this can have positive impact (i.e. tuned circuit in radio or T.V. receiver responds strongly to waves with frequencies close to its resonant frequency, this fact is used to select a particular station when rejecting the others).

The simple pendulum

- idealized situation
- mass m suspended by massless wire of length L.



Restoring force:

$$G_t = -G \sin\theta$$

provided by gravity.

for small angles: $\sin\theta \simeq \theta \simeq \frac{x}{L}$

$$\Rightarrow F_t = -mg\theta = -\underbrace{mg\frac{x}{L}}_{k} \qquad \boxed{k = \frac{mg}{L}}$$

$$= -kx$$

One can identify then $\omega = \sqrt{\frac{k}{m}}$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{g}{L}}}$$

$$\Rightarrow \quad T = \frac{2\pi}{\omega} \quad ; \quad \boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

$$f = \frac{\omega}{2\pi} \qquad \boxed{f = \frac{1}{2\pi}\sqrt{\frac{L}{g}}}$$

simple pendulum, small amplitude