DYNAMICS OF ROTATIONAL MOTION

TORQUE, ANGULAR MOMENTUM. CONSERVATION LAWS

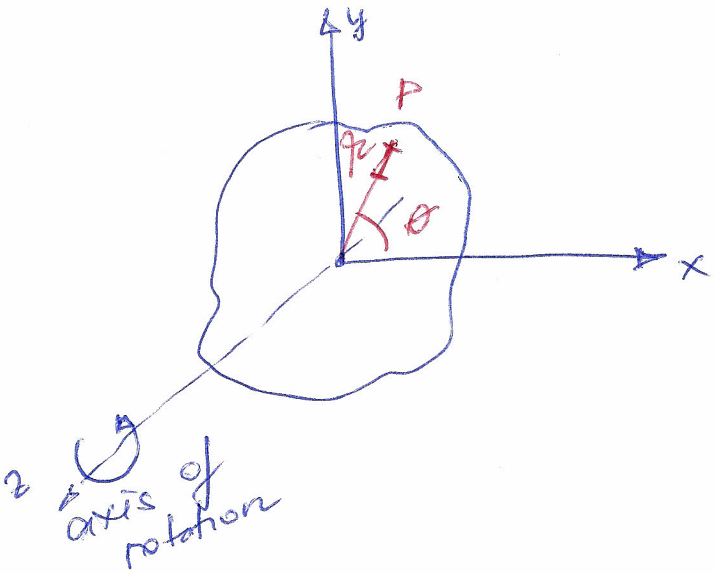
[1] ROTATION OF RIGID BODY, KINEMATICS

- beyond the point approach
- size and shape of the body counts
- we neglect the deformation

} ⇒ rigid body approach.

Case of rotation around an axis, stationary in an inertial frame of reference.

① Angular velocity and acceleration



P – particular point of the body; described by $(\vec{r}, \theta)$

θ – coordinate for rotation

Average angular velocity

$$\omega_{av-z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

ratio of the angular displacement

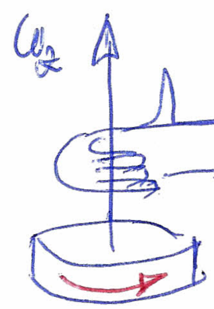
Instantaneous angular velocity

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$

$$[\omega]_{SI} = \frac{\text{rad}}{s}$$

Angular velocity as a vector

→ the $\vec{\omega_z}$ direction is along the rotation axis given by the right-hand rule



Angular acceleration

average: $\alpha_{av-z} = \frac{\Delta \omega_z}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$

instantaneous: $\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta \omega_z}{\Delta t} = \frac{d\omega_z}{dt}$

$\underline{\underline{\text{obs}}}$: $[\alpha_z] = \frac{\text{rad}}{s^2}$;

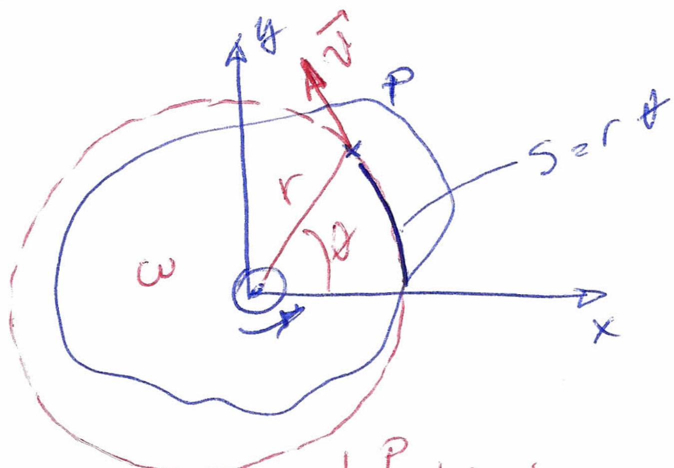
$$\boxed{\alpha_z = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2}}$$

Rotation with constant angular acceleration

- one can deduce similar mot. laws as for straight line motion with constant acceleration

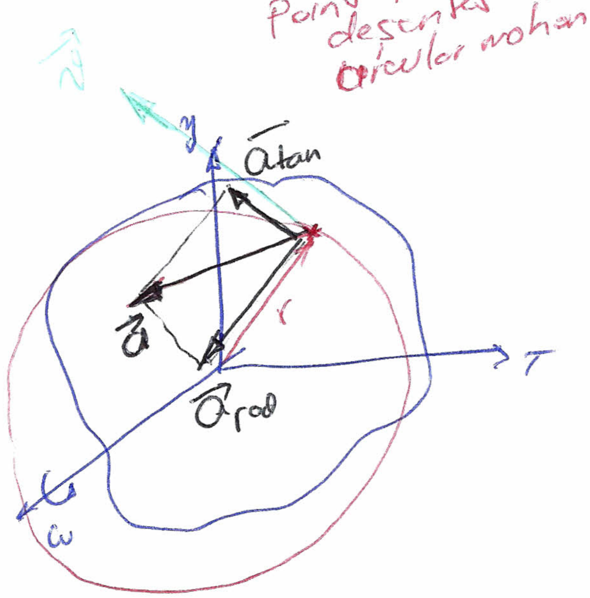
Rotation around fixed axis	Straight line motion
$\alpha_z = ct$	$a_x = ct$
$\omega_z = \omega_{0z} + \alpha_z t$	$v_x = v_{0x} + a_x t$
$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_z t^2$	$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$
$\omega_z^2 = \omega_0^2 + 2\alpha_z(\theta - \theta_0)$	$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$

Relating linear and rotational kinematics



$$\left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right|$$

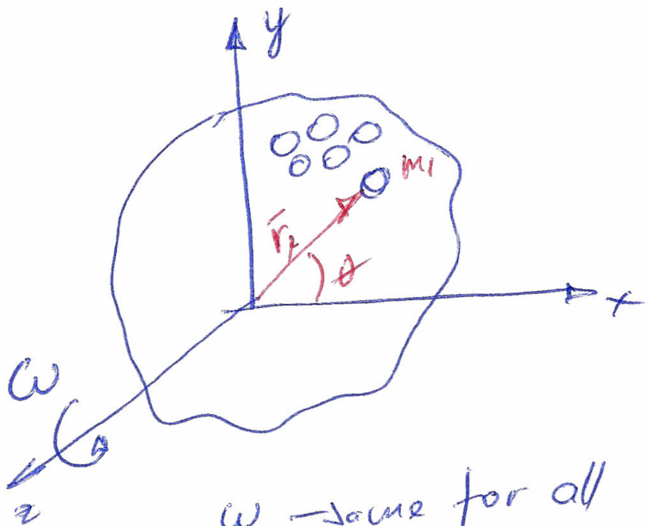
$$\Rightarrow \boxed{v = r\omega}$$



$$\boxed{\begin{aligned} a_{tan} &= \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \\ a_{rad} &= \frac{v^2}{r} = \omega^2 r \end{aligned}}$$

$$\vec{a} = \vec{a}_{rad} + \vec{a}_{tan}$$

Energy in the rotational motion



rigid body $m = \sum m_i$

$v_i = r_i \omega$

$\Rightarrow K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \omega^2 r_i$

ω – same for all m_i

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i \omega^2 r_i^2 = \frac{1}{2} \underbrace{\left(\sum_i m_i r_i^2\right)}_{} \omega^2$$

$$\boxed{I = \sum_i m_i r_i^2}$$

moment of inertia of the body

$$\Rightarrow \boxed{K = \frac{1}{2} I \omega^2}$$

rotational kinetic energy of a rigid body.

Obs: Interpretation of I (moment of inertia)

The larger is I, the larger is the kinetic energy of a rotating rigid body with a given angular speed ω

$\Longrightarrow$ Larger is I, the most difficult is to stop a rotating rigid $\Rightarrow$ rotational inertia

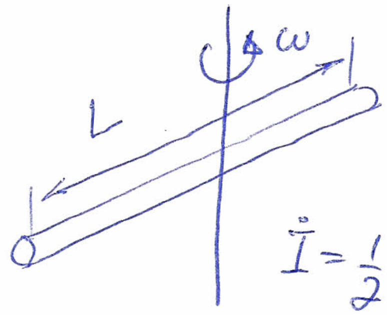
$\Delta K = W$

General calculation formalism

$$I = \sum m_i r_i^2 \xrightarrow[\text{continuous}]{} \int r^2 dm = \rho \int r^2 dV$$

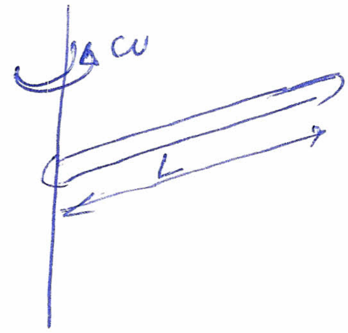
$dm = \rho dV$

The moment of inertia depends on the:
- shape of the body
- how the rotation axis is placed.



$I = \frac{1}{2} ML^2$

$\neq$



$I = \frac{1}{3} mL^2$

Paralel axis theorem

A body doesn't have only one moment of inertia, it has infinitelly many because there are many axes around which it can rotate.

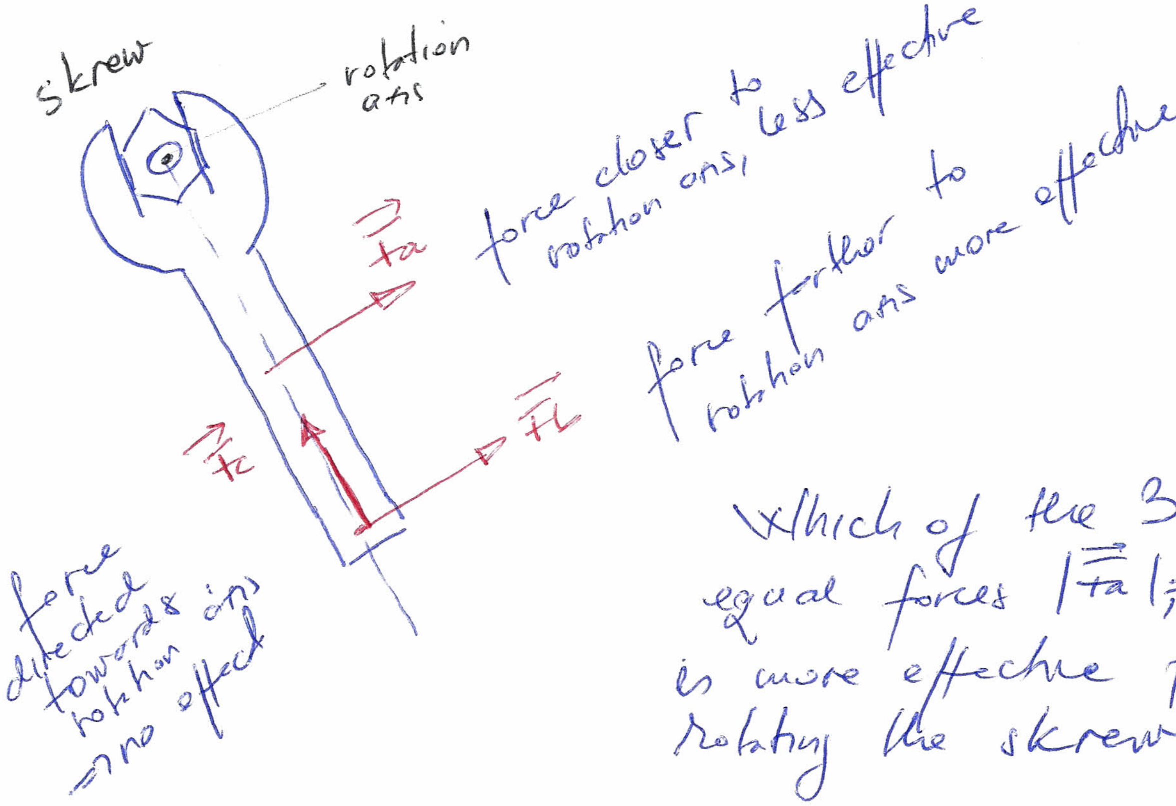
There is a paralel axis theorem that gives the relationship between the moment of inertia I_{cm} of a body of mass M about an axis through the center of the mass and the moment of inertia of any axis parallel to the original one but displaced with a distance d.

$$I_P = I_{cm} + Md^2$$

DYNAMICS OF ROTATIONAL MOVEMENT

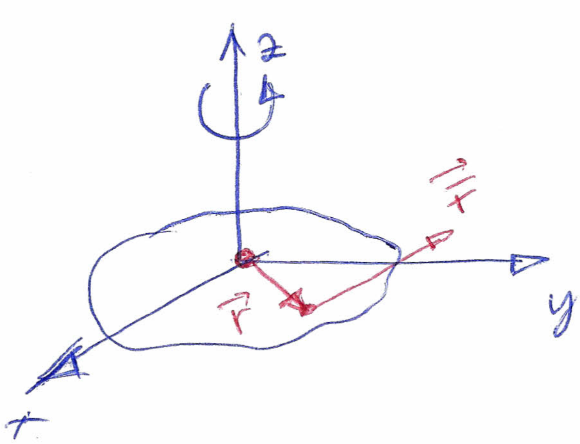
① Torque

– this new physical quantity describe the twisting or turning effort of a force ⟹ the net torque acting on a rigid body determine its angular acceleration in the same way that a net force on a body determines its linear acceleration.



Which of the 3 equal forces $|\vec{F_a}| = |\vec{F_b}| = |\vec{F_c}|$ is more effective for rotating the skrew?

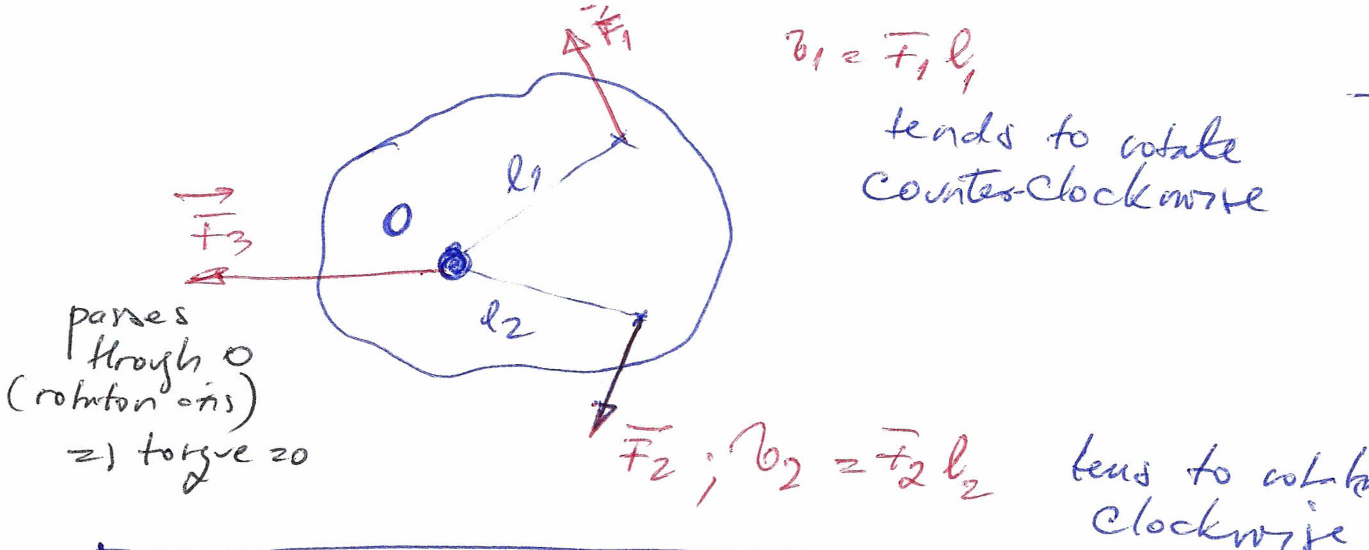
⟹ Torque as a vector



$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

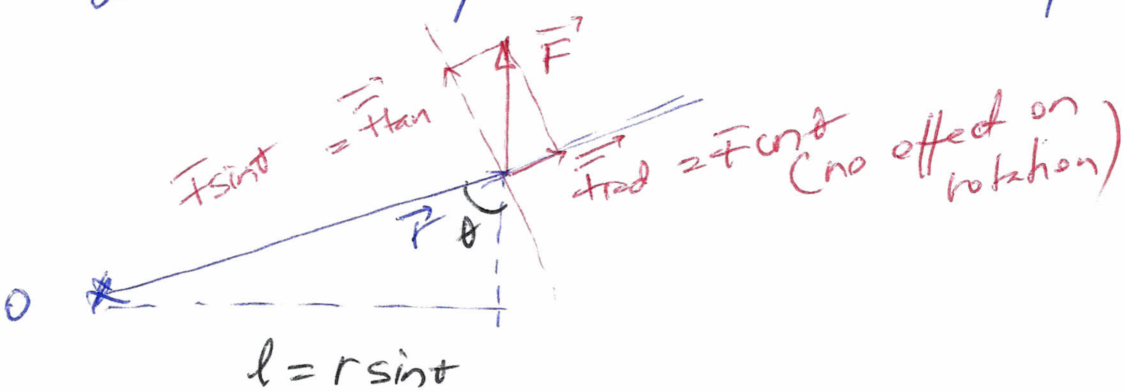
direction of $\vec{\tau}$

⟹ RIGHT HAND RULE



$$\boxed{|\tau| = |\vec{r} \times \vec{F}| = rF\sin(\vec{r}, \vec{F})}$$

Torque is always measured about a point



=> 3 ways to define torque:

$$\boxed{\tau = Fr\sin\theta = Fl = F_{tan}\, r}$$

② Torque and angular acceleration

Consider a rigid body constituted by a large number of particles, rotating around z axis.

The net force acting on m_i is:

$$\vec{F}_i = \underbrace{\vec{F}^i_{tan}}_{\text{torque}} + \underbrace{\vec{F}^i_{rad}}_{\text{no torque}}$$

Newton 2nd law

$$\Rightarrow \quad r_i \;\Big|\; F_{i\,tan} = m_i\, a_{i\,tan}$$

$$a_{i\,tan} = r_i\, \alpha_z$$

$$\Rightarrow \underbrace{r_i F_{i\,tan}}_{\text{torque}} = m_i r_i^2 \alpha_z$$

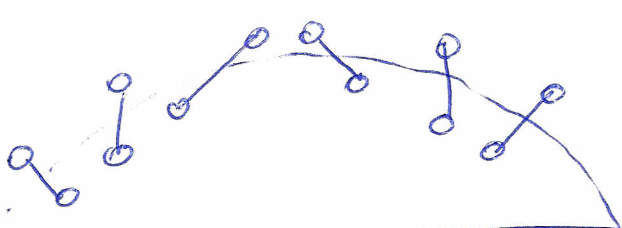
$$\tau_{i_z} = m_i r_i^2 \alpha_z \Rightarrow$$

$$\sum_i \tau_{i_z} = \underbrace{\left(\sum_i m_i r_i^2\right)}_{I} \alpha_z \Rightarrow \boxed{\sum_i \tau_{i_z} = I \alpha_z}$$

rotational analog of the Newton's 2nd law

Moving of rigid body

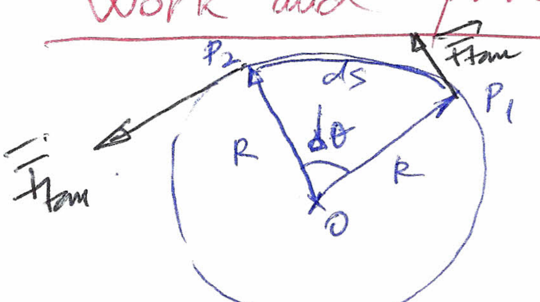
combined translation + rotation



$$\boxed{K = \underbrace{\tfrac{1}{2} M V_{cm}^2}_{\text{translational}} + \underbrace{\tfrac{1}{2} I_{cm}\, \omega^2}_{\text{rotational}}}$$

KINETIC ENERGY

Work and power in rotational motion



The work done by $\vec{F}_{tan}$ when point P moves from P_1 to P_2 with ds $\Rightarrow$

$$dW = F_{tan}\, dS = \underbrace{F_{tan} R}_{\tau}\, d\theta = \tau\, d\theta$$

$$\Rightarrow \boxed{W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta}$$

if τ = constant with $\theta \Rightarrow \boxed{W = \tau \Delta\theta}$

Work-energy theorem

$$W = \Delta K = K_2 - K_1 \Rightarrow$$

$$\boxed{W = \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2}$$

(discuss analogy with translational motion)

Power

$$P = \frac{dW}{dt} = \frac{d}{dt}(\tau\, d\theta) = \tau \frac{d\theta}{dt} = \tau\,\omega$$

$$\Rightarrow \boxed{P = \tau\,\omega}$$

(analogous of $P = F v$)

Angular momentum

Every rotational quantity defined for rotational motion of a rigid body has an analog in the translational motion and vice-versa.

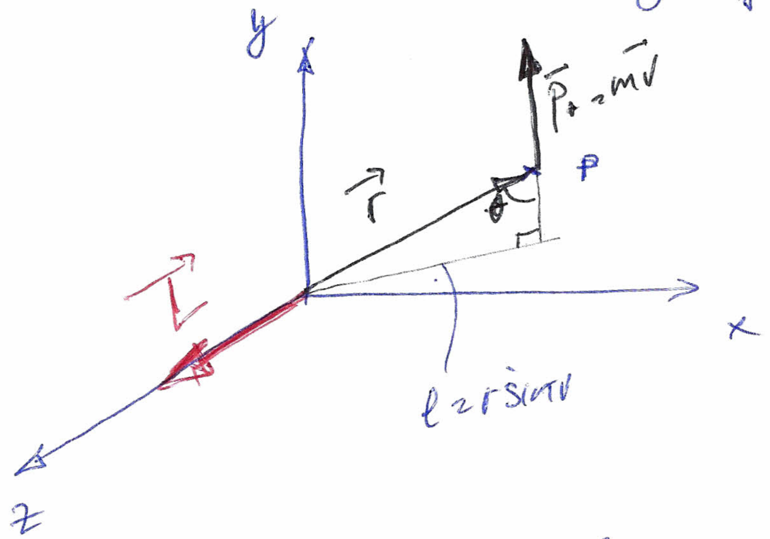
The analog of the momentum of particle is the angular momentum

→ particle linear momentum $\vec{p} = m\vec{v}$

⇒ angular momentum of a particle:

$$\boxed{\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}}$$

$$[L] = m \cdot kg \cdot m/s = kg \cdot \frac{m^2}{s}$$



$L = mvr\sin\theta = mvl$

When a force acts on a particle:
- velocity
⇒ momentum change } ⇒ angular momentum changes

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \underbrace{\frac{d\vec{r}}{dt} \times m\vec{v}}_{(\vec{v} \times m\vec{v} = 0)} + \vec{r} \times \frac{d}{dt}(m\vec{v})$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times \underbrace{m\frac{d\vec{v}}{dt}}_{\vec{F}} = \vec{r} \times \vec{F} = \vec{\tau}$$

L ⇒ eq.
2nd principle

$$\Rightarrow \boxed{\frac{d\vec{L}}{dt} = \vec{\tau} = \sum \vec{\tau}_i}$$

The rate of change of angular momentum of the particle equals the torque of the net force

Angular momentum of a rotating rigid body

$m_i, \ldots \qquad v_i = r_i \omega$

$$L_i = r_i m_i v_i = m_i r_i^2 \omega$$

$$L = \sum L_i = \left(\sum m_i r_i^2\right)\omega = I\omega$$

Vectorial : $\boxed{\vec{L} = I\vec{\omega}}$

$$\boxed{\frac{d\vec{L}}{dt} = \sum \vec{\tau} = I\frac{d\omega}{dt} = I\vec{\alpha}_z}$$

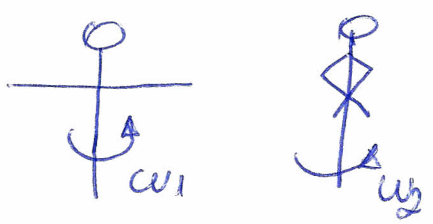
Conservation of angular momentum

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \; ; \quad \text{if} \quad \sum \vec{\tau} = 0 \Rightarrow$$

$$\frac{d\vec{L}}{dt} = 0 \Rightarrow \boxed{\vec{L} = \text{constant}}$$

when the net ~~external force~~ torque is zero, the total ANGULAR MOMENTUM of the system is constant.

ex : skater turning around with open/closed arms



$$L = ct \Rightarrow I_1\omega_1 = I_2\omega_2$$

$$\omega_2 = \omega_1 \frac{I_1}{I_2} \; ; \quad I_1 > I_2 \Rightarrow$$

$\boxed{\omega_2 > \omega_1}$ the angular speed increases when closing arms.

EQUILLIBRIUM AND ELASTICITY

A body that can be modeled as a particle is in equillibrium whenever the vector sum of all forces acting on it is zero.

This condition is not enough for extended body. Here, ~~points~~ forces acting on different points exerts a torque.

⟹ additional requirement to avoid rotation ⟹ $\boxed{\text{sum of torques} = 0}$

Condition of equillibrium (static / dynamic equillibrium)

(1) $\boxed{\sum \vec{F} = 0 \qquad \sum F_x = 0, \quad \sum F_y = 0; \quad \sum F_z = 0}$

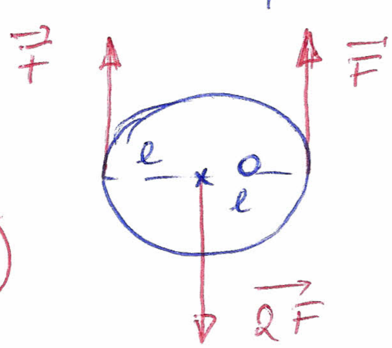
first condition of equillibrium (the only condition for particle)

(2) $\boxed{\sum \vec{\tau} = 0}$ about any point

second condition of equillibrium

Ex: To be in static equillibrium, a body has to fulfill both conditions

(1)

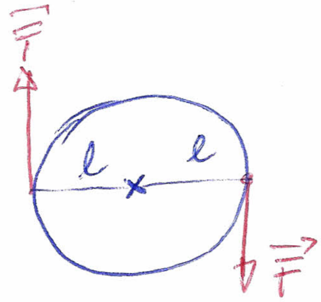


1st condition satisfied $\sum F = F + F - 2F = 0$

2nd condition satisfied $\sum \vec{\tau} = Fl - Fl \pm 0 = 0$

⟹ body in static equillibrium

2



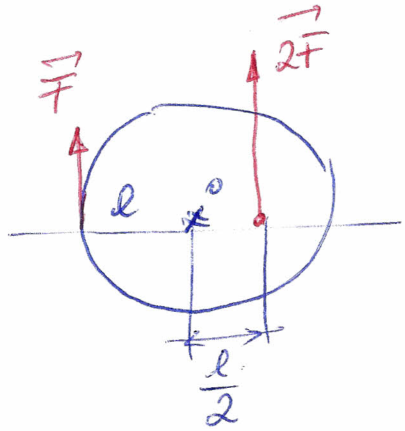
1st cond. satisfied $\sum \vec{F} = F - F = 0$

$\Rightarrow$ no translational movement.

2nd cond NOT SATISFIED : $\sum \vec{\tau} = Fl + Fl = 2Fl$

$\Rightarrow$ rotation clockwise

3

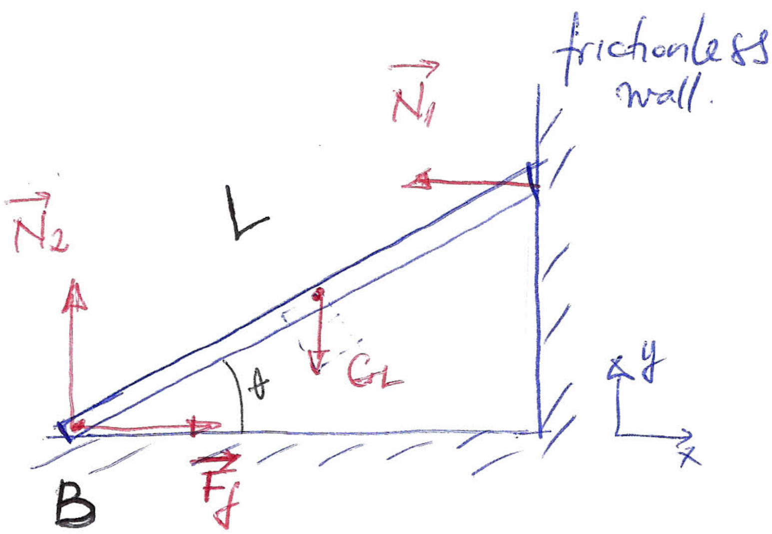


1st condition NOT SATISFIED $\sum \vec{F} = F + 2F = 3F \neq 0$

second condition satisfied: $\sum \vec{F} = Fl - 2F\,l/2 = 0$

$\Rightarrow$ translational movement

Solving rigid body equillibrium problems



$$\begin{cases} \sum \vec{F} = 0 \ ; \quad \sum F_x = 0, \ \sum F_y = 0 \\ \sum \vec{\tau} = 0 \end{cases}$$

L – ladder length
θ – angle ladder/ground
M – ladder mass

$$\sum F_x = F_f - N_1 = 0$$
$$\sum F_y = N_2 - Mg = 0$$

; forces

(1), (2)

The torque equillibrium equation for point B

$$\boxed{\vec{\tau}_B = N_1 l \sin\theta - Mg \, l/2 \, \cancel{\cos\theta} = 0} \quad (3)$$

(1), (2), (3) set of equations which can be solved for any unknown.

$$N_1 = F_f = \mu N_2 = \mu Mg$$
$$N_2 = Mg$$

$$(3) \Rightarrow \tan\theta = \frac{Mg \, l/2}{N_1 l} = \frac{Mg}{2\mu Mg} = \frac{1}{2\mu}$$

$$\boxed{\tan\theta = \frac{1}{2\mu}}$$

If $\theta < \theta_c \Rightarrow$ the ladder slides....

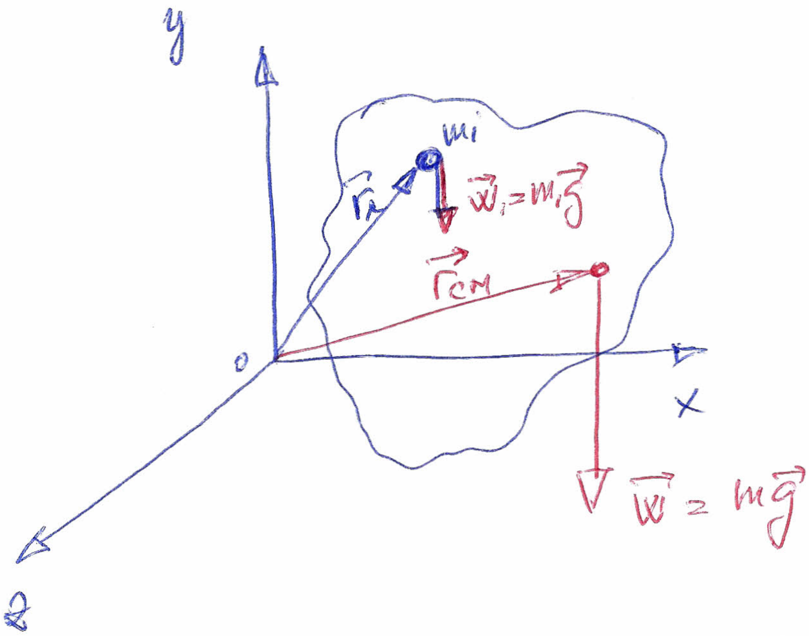
Center of gravity

In the most equillibrium problems one of the forces acting on the body is the weight. We need to calculate the torque of this force. The weight does not act at a single point, it is destributed over the whole body.

But, we can calculate the torque due to the body weight, assuming that the entire weight is concentrated at a point called the center of gravity.

If we neglect the variation of $\vec{g}$ with the vertical dimensions of the body, the center of gravity is identical with the center of mass.

$$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$



$m = \sum m_i$

$$\vec{\tau} = \sum \vec{\tau}_i = \sum_i m_i \vec{r}_i \times \vec{g} \qquad \Big| \frac{\sum m_i}{\sum m_i}$$

$$\Rightarrow \vec{\tau} = \sum m_i \left(\frac{\sum m_i \vec{r}_i}{\sum m_i} \right) \times \vec{g} = \underline{\vec{r}_{CM} \times M\vec{g}}$$

Finding and using the center of gravity

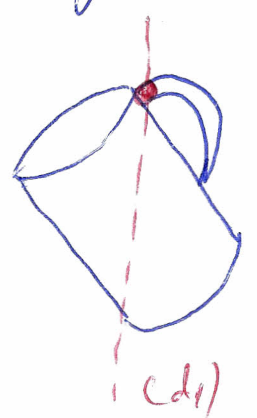
→ for regular shape and symmetric bodies with homogeneous mass distribution, the CG is the geometric center (ex. cube, sphere, rectangular plate, ...)

Finding the CG of irregurally shaped body.

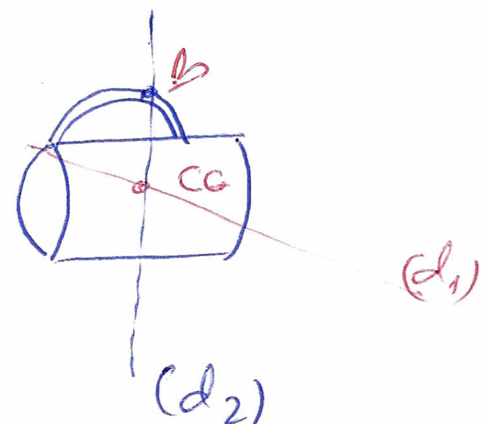
When a body acted by gravity is suspended at a single point, the CG is always on the vertical axis going through that point (if elsewhere, the weight would exert a torque which would lead to out-of-equilibrium position).

Using this reasoning, if we suspend that body at different points, the CG will be at the crossing point between the different vertical axes.

ex mug



suspend the mug at point A; the line (d1) passes CG.



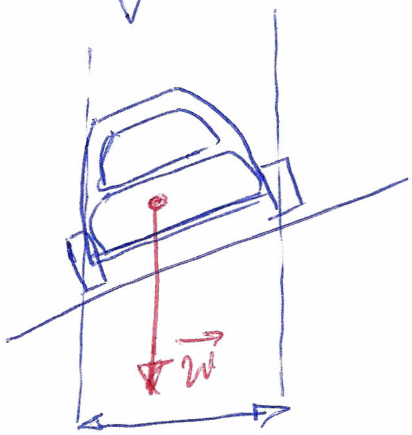
suspending in other point B → vertical direction (d2)

$\boxed{\text{CG} \in \text{intersection of (d1) and (d2)}}$

→ valid for any complex shape-body.

Position of CG and equilibrium against overturn

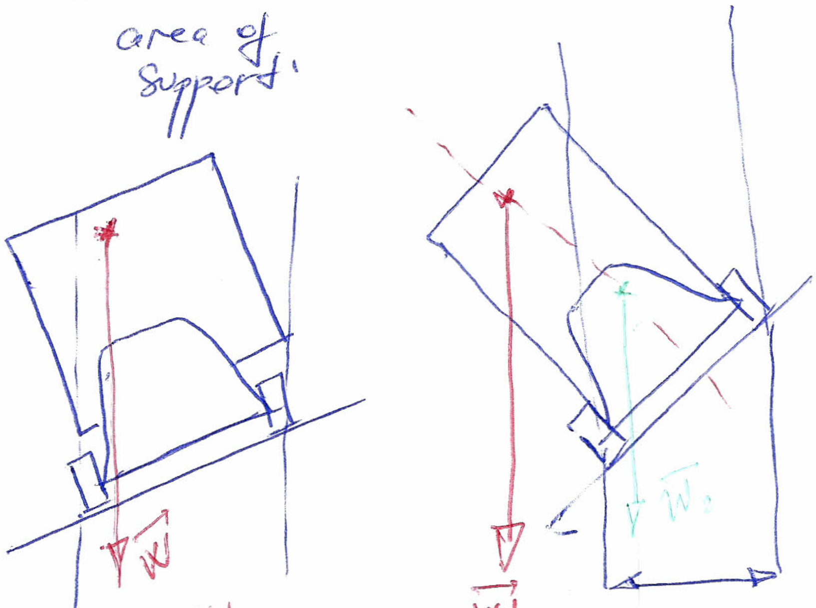
(1)



area of support

If the CG lies within the area of support, the car (body) is in equilibrium

(2)



equilibrium

the higher the CG, the smaller the incline needed to overturn the vehicle

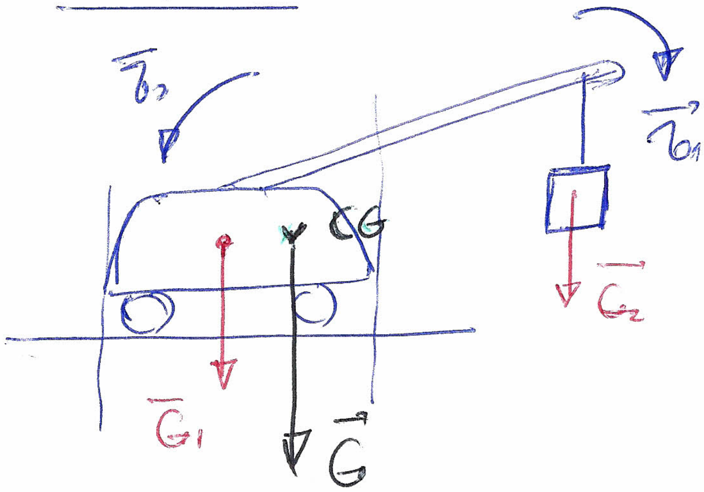
center of gravity outside of support area $\Rightarrow$ vehicle tips over.

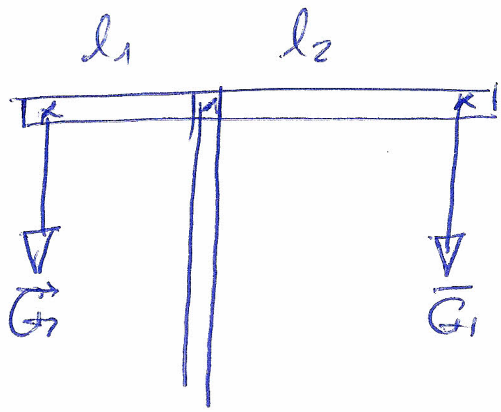
The lower the CG and the larger the area of support the more difficult ~~to~~ to overturn a body

→ four legs animals are stable, naturally

→ two legs animals (humans, birds) need relatively large feet to give them a reasonable area of support.
For horizontal body two-legs animals (chicken, dynosaurs) the equilibrium is controlled by balancing the head and/or the tail).

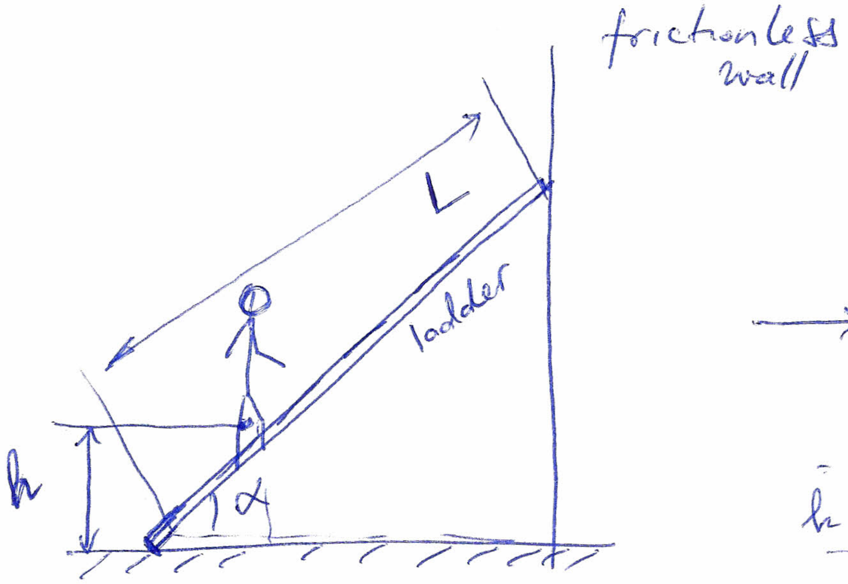
Cranes

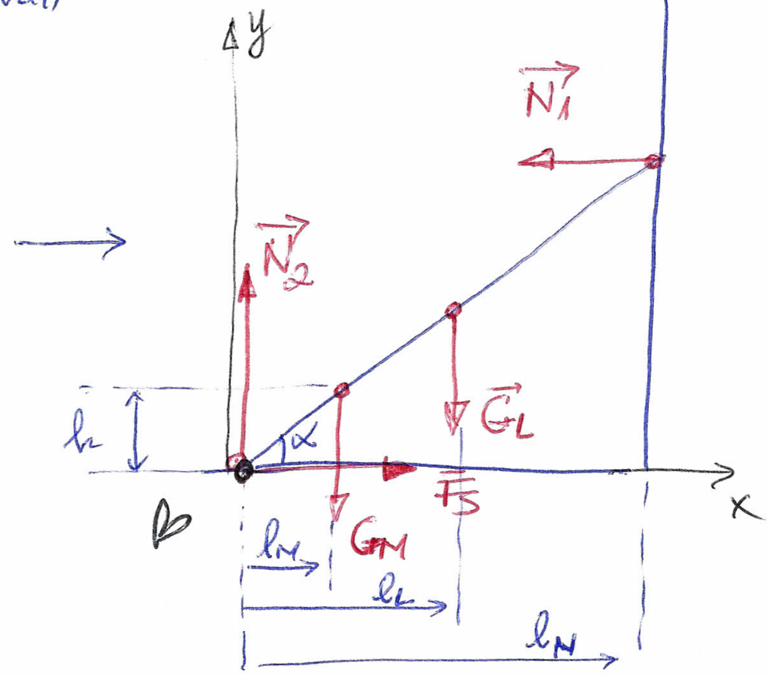




$$G_1 l_1 = G_2 l_2$$

Solving equilibrium problems





Static equilibrium

$$\sum F_x = 0 \Rightarrow F_S - N_1 = 0 \quad (1)$$

$$\sum F_y = 0 \Rightarrow N_2 - G_M - G_L = 0 \quad (2)$$

Rotation equilibrium

$$\sum \vec{\tau}_B = 0$$

(sum of torques for rotation around an axis going through point B).

$$\vec{\tau}_B = G_M l_M + G_L \cdot l_L - N_1 l_N$$

the traces of the three forces are:

$$l_M = \frac{h}{\tan\alpha} ; = \frac{h\cos\alpha}{\sin\alpha}$$

$$l_L = \frac{L}{2}\cos\alpha \quad , \quad l_N = L\cos\alpha$$

$\Rightarrow$

$$G_M h \frac{\cancel{\cos\alpha}}{\sin\alpha} + \frac{L}{2} G_L \cancel{\cos\alpha} - N_1 L \cancel{\cos\alpha} = 0$$

$$\Rightarrow \quad \frac{G_M h}{\sin\alpha} + \frac{L}{2} G_L - N_1 L = 0 \qquad (3)$$

$$\begin{cases} F_S - N_1 = 0 \\ N_2 - G_M - G_L = 0 \\ \dfrac{G_M h}{\sin\alpha} + \dfrac{L}{2} G_L - N_1 L = 0 \end{cases}$$

3 equations with 3 unknowns N_1, N_2, F_S.

obs: $G_M = Mg$ (weight of man)
$G_L = mg$ (weight of ladder)

The minimum coefficient for the static friction needed to prevent slipping at the base.

$$F_S \le \mu_S N_2 \Rightarrow \boxed{(\mu_S)_{min} = \frac{F_S}{N_2}}$$

STRESS, STRAIN, ELASTIC MODULI

⇒ beyond the rigid body approach

Stress = characterizes the strength of the forces causing deformation

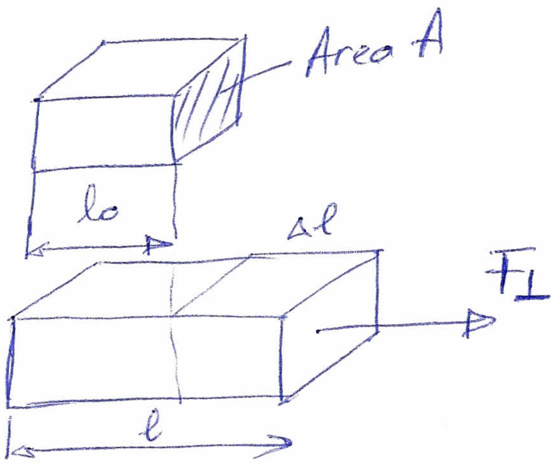
Strain = defines the quantity of deformation.

The two are proportional with a proportionality constant called elastic modulus.

$$\boxed{\frac{\text{Stress}}{\text{strain}} = \text{Elastic modulus}}$$

(Hooke's law)

Tensile and compressive stress and strain



$$\boxed{\text{Tensile stress} = \frac{F_\perp}{A}}$$

scalar quantity because $F_\perp$ is the force magnitude

$[\text{Tensile stress}] = \frac{N}{m^2} = Pa$

same as for pressure

$$\boxed{\text{Tensile strain} = \frac{\Delta l}{l_0} = \frac{l - l_0}{l_0}}$$

dimensionless.

For sufficiently small tensile stress, the stress and strain are proportional via the Young's modulus

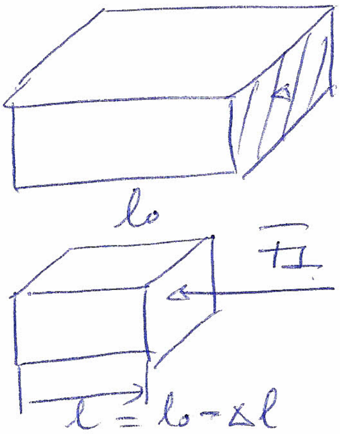
$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp / A}{\Delta l / l_0} = \frac{F_\perp}{A} \Big/ \frac{\Delta l}{l_0}$$

$$\Rightarrow \boxed{\frac{F_\perp}{A} = Y \frac{\Delta l}{l_0}}$$

Y = material dependent.

$Y \uparrow \Rightarrow$ material hardly stretchable

Similar analysis for compression



$l = l_0 - \Delta l$

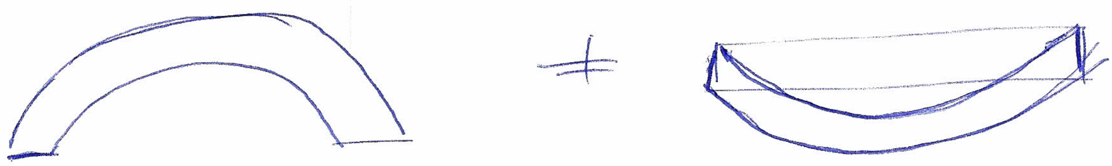
$\Rightarrow$ compressive stress
compressive strain.

Similar expression as for tensile stress, strain, Young modulus.

For many materials the Young modulus has the same value for tensile and compressive stress.

However, composite materials (ex. concrete, stone) are different. Stones were the first building elements for ancient civilisations (Babylonians, Asyrians, Romans). Their building were designed to avoid <u>tensile stresses</u>

$\Rightarrow$ they used arches in doorways and bridges where the weight of the material compresses the stone together and does not place them over tensile stress.



$\Rightarrow$ today we <u>reinforce concrete by iron</u> (1853 COIGNET)
$\Rightarrow$ high toleration to tensile stress

Bulk stress and strain

When stress is an uniform pressure on all sides and the resulting deformation is a volume change (i.e. body plounged into a fluid - water).

$\Rightarrow$ bulk stress (volume stress) $\longrightarrow$ bulk strain (volume strain)

If an object is immersed in a fluid at rest, the fluid exerts a force on any part of the object's surface; this force is perpendicular to the surface.

$\Rightarrow$ pressure = force $\perp$ to the unit area

$$\boxed{p = \frac{F_\perp}{A}}$$ pressure in a fluid

Pressure plays a role of stress in the volume deformation; the correspondent strain is the fractional change in volume.

$$\text{Bulk (volume) strain} = \frac{\Delta V}{V_0}$$

$\Rightarrow$ Hooke's law:

$$\boxed{B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V / V_0}}$$ bulk modulus

[modul de compresibilitate]

"$-$" sign means that the increase of the pressure always determines the decrease of the volume.

$\Delta V > 0 \Rightarrow \Delta p < 0$

$\Delta p > 0 \Rightarrow \Delta V < 0$; B = always positive.

The reciprocal of the bulk modulus is called <u>compressibility</u> and it is denoted by k

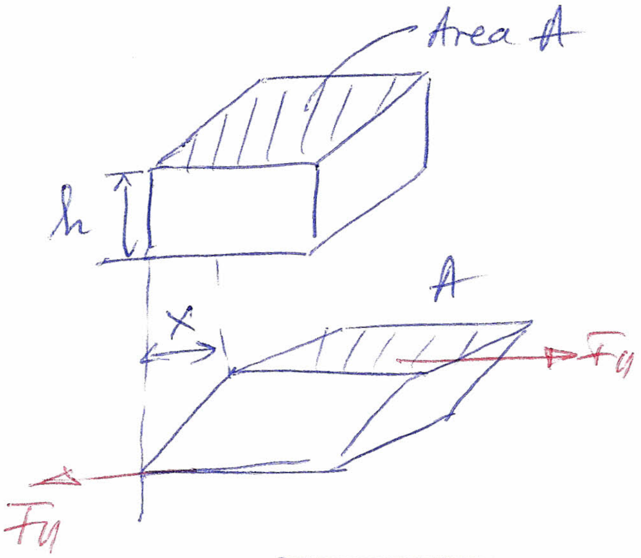
$$\boxed{k = \frac{1}{B} = -\frac{\Delta V / V_0}{\Delta p} = -\frac{1}{V_0}\frac{\Delta V}{\Delta p}}$$

= fractional decrease in volume $-\frac{\Delta V}{V_0}$ per unit increase Δp in pressure

$$[k] = [Pa]^{-1} \text{ or } [atm]^{-1}$$

Materials with <u>small bulk modulus</u> and large compressibility are <u>easy to compress</u>.

Shear stress / strain



$$\text{shear stress} = \frac{F_\parallel}{A}$$

$$\text{shear strain} = \frac{x}{h}$$

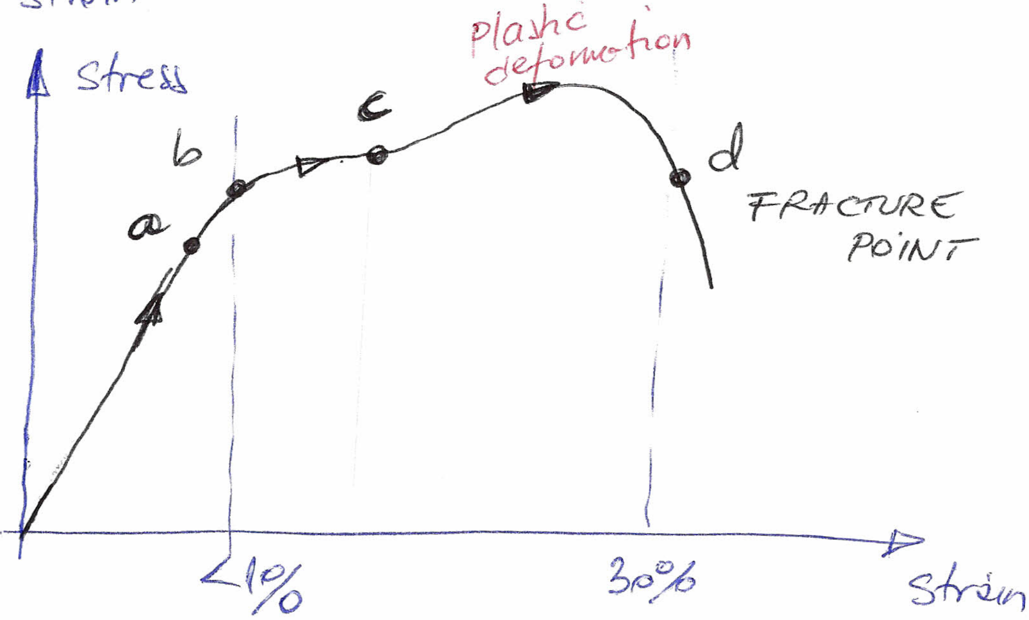
The corresponding elastic modulus is called shear modulus S

$$\boxed{S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F_\parallel / A}{x/h} = \frac{F_\parallel}{A} \cdot \frac{h}{x}}$$

Elasticity and plasticity

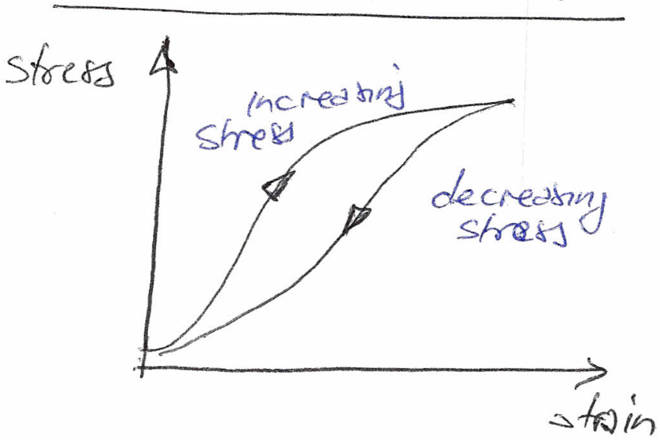
Hook's law — the proportionality of stress and strain has a limited range of validity.

Suppose we plot a graph of stress as a function of strain



- a – proportionality limit
- [a-b] → beyond proportionality limit; Hooke's law not obeyed.
  → deformation still reversible; if stress is removed the material returns to its original shape.
  the forces are conservative and the work put to dilate is equal to the work recovered when the stress is removed. ⇒
  zero total work ⇒ ellastic regime
- Beyond point b ⇒ plastic deformation (plastic flow)
  when the stress is removed the initial shape does not recovered.
  ⇒ irreversible
- d – fracture point
  → ductile materials : large b-d zone
  → BRITTLE materials ("fragil") : point d close to the ellastic zone

Ellastic HYSTERESYS



due to nonconservative forces related to external friction when stretching and compressing (ex. spring) the work is not the same
⇒ hysteresis / dissipation of energy

ex: Rubber with large ellastic hysteresis is very good to absorb vibrations

Breaking stress ⇒ stress required to cause fracture of a material.

Two materials (ex steels) can have equal ellastic constants but different breaking stress.