Superposition of waves
(harmonical oscillations)

In the course you have studied:

→ the interference of waves with same frequency and different or equal amplitude (see also last seminary)

(1) **BEATS** → the interference of two waves with slightly different frequency $\omega_1 \neq \omega_2$ and equal amplitude ⟹ the beats phenomena

$$y_1 = A\cos(kx + \omega_1 t) \longleftarrow \qquad \neq A\cos\omega_1 t$$
$$y_2 = A\cos(kx - \omega_2 t) \longrightarrow \qquad \neq A\cos\omega_2 t$$

Propagating direction

interfere at $x=0$

$$y = y_1 + y_2 = A\left[\cos\omega_1 t + \cos\omega_2 t\right]$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\left(\frac{A-B}{2}\right)$$

$$\Rightarrow y = 2A\cos\frac{(\omega_1+\omega_2)t}{2}\cos\frac{(\omega_1-\omega_2)t}{2}$$

$$\omega = 2\pi f$$

$$y = 2A\cos 2\pi\frac{(f_1+f_2)}{2}t\cos 2\pi\frac{(f_1-f_2)}{2}t$$

Resulting wave = double amplitude · high frequency modulation · low frequency modulation

⟹ <u>beats</u> = amplifications and atenuation of wave amplitude (intensity $\sim A^2$)

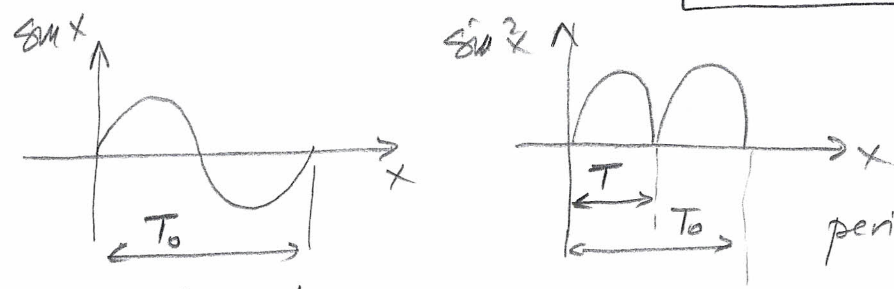
The amplitude variation causes variation of loudness called beats

The amplitude factor:

$$A = 2A \sin 2\pi \frac{(f_1 - f_2)}{2} t$$ produces a variation

of intensity $I \sim A^2$, and the frequency of

$$\sin^2\left(2\pi \frac{(f_1-f_2)}{2} t\right) \text{ is } \boxed{f_{beat} = 2\frac{(f_1-f_2)}{2} = f_1 - f_2}$$



period = $\frac{\text{sinus period}}{2} = \frac{T_0}{2}$

$f_0 = \frac{1}{T_0}$ $\Rightarrow f = 2 f_0$

(2) LISSAJOUS FIGURES

We focus now on interference of harmonical oscillations (or waves) that are perpendicular and have different pulsations ω

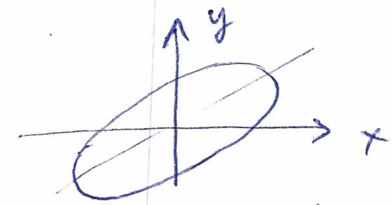
2.1. Oscillations of same pulsation ω

Consider a point M submitted simultaneously to two harmonical oscillations with same frequency ω along Ox and Oy direction of a Cartesian system xOy.

$$x = A \sin \omega t$$
$$y = B \sin(\omega t + \varphi)$$

Based on $\sin^2 \omega t + \cos^2 \omega t = 1$, eliminating the time in the above equations, one gets:

$$\boxed{\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos\varphi = \sin^2\varphi} \qquad (1)$$



Obs: $\sin(A+B) = \sin A \cos B + \cos A \sin B$

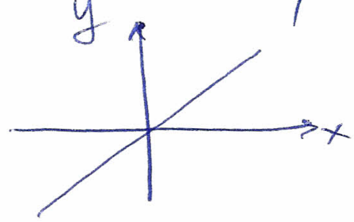
The equation (1) is the equation of an ellipse, so that the resultant of two harmonical oscillations of same frequency and perpendicular oscillation direction is an eliptical oscillation.

We will analyse some particular cases, as a function of the phaseshift φ between the two oscillations:

1. the phase difference $\boxed{\varphi = 2n\pi}$ $(n = 0, 1, 2, \ldots)$ the oscillations are in-phase $\Rightarrow$

$$x = A \sin \omega t$$
$$y = B \sin \omega t \quad \Rightarrow \quad \boxed{y = \frac{B}{A} x}$$

this is the equation of a straight line passing through the origin with the slope $\frac{B}{A}$ (ellipse degenerated into two confounded lines)

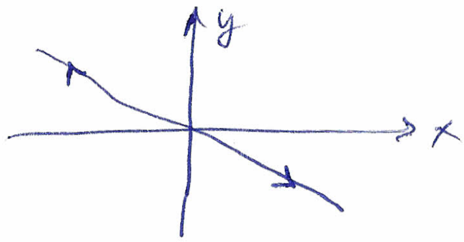


The interference of two perpendicular oscillations with same frequency and in-phase ($\varphi = 0$) leads to an harmonic oscillation along a linear direction

The reciprocal is also valid: any linear harmonic oscillation can be decomposed in two perpendicular harmonic motions.

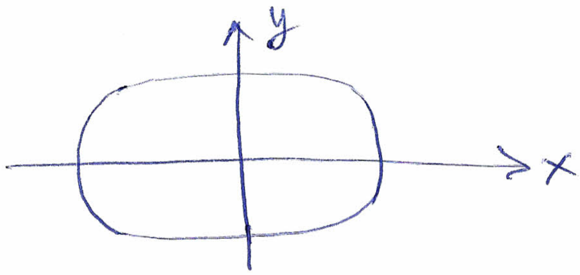
(2) the phase difference is $\varphi = (2n+1)\pi$ $n = 0, 1, 2, \ldots$

$$\Rightarrow \boxed{y = -\frac{B}{A}x}$$ straight line equation, too



(3) the phase difference $\varphi = (2n+1)\pi/2$

$$\Rightarrow \boxed{\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1}$$ ellipse equation whose axes concide with the axes of x and y



In the particular case when $A = B$ (equal amplitude) $\Rightarrow \boxed{x^2 + y^2 = A^2}$ equation of a circle of radius A

<u>Consequence</u>:

Any circular motion can be decomposed into two linear harmonical motions of equal amplitudes and difference of phase $\varphi = (2n+1)\pi/2$, the oscillations being perpendicular one to the other.

2.2. Oscillations with different frequencies ω

$$x = A \sin \omega_x t$$

$$y = B \sin(\omega_y t + \varphi)$$

In this case, the resulting trajectory has a more complex shape.

In the situation when the ratios of frequencies is a rational fraction

$$\frac{\omega_y}{\omega_x} = \frac{f_y}{f_x} = \frac{n_x}{n_y}$$ where n_x, n_y are integers

the trajectories are called LISSAJOUS FIGURES

A Lissajous figure crosses n_x times each of the rectangle axis parallel to OX and n_y times the rectangle axis paralel to oy. Thee orientation of the Lissajous figures depend on the phase difference of the component oscillations.

In case when the ratio of frequencies is not a rational fraction: $\frac{\omega_y}{\omega_x} \neq \frac{n_x}{n_y}$ the curves are no more closed and Lissajous figures not observed

We represent some Lissajous figures of two perpendicular oscillations with same amplitude and phase difference of $0, \pi/4, \pi/2, 3\pi/4, \pi$ in case when the frequency ratios are $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}$.

Experimental analysis.

(*)
One uses the software SOUNDCARD OSCILLOSCOPE

- We are going to plot X-Y graph (chose from menu)
- Using the menu: Signal Generator and chose separate window you will get a two channel signal generator. Here you can activate one by one the channels and chose: for each channel

→ signal amplitude
→ " frequency

Then, you will send to the scope CH1 and CH2

CH2 → on y
CH1 → on x of the X-Y Graph.

⚠ On settings menu you should first chose the Windows sound parameters:

Audio devices

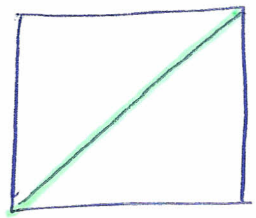
Output : Loudspeakers (or Scope loopback)
Input : Scope Loopback

Obs : If you implicitly leave Input from Loudspeakers or Microphone, your analysis will be disturbed by external noise.

Parameters

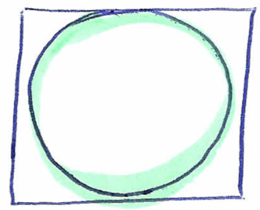
(1) Similar amplitudes $A_1 = A_2 = 0{,}5$
similar frequencies $f_x = f_y = 100\,Hz$

Phaseshift: $\varphi = 0$

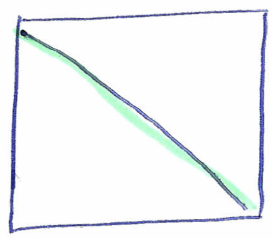


xy-Graph

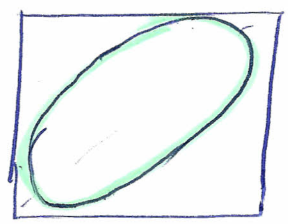
$\varphi = \pi/2$



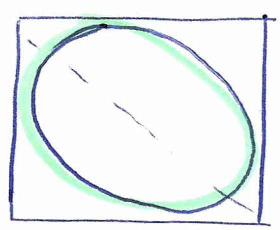
$\varphi = \pi$



$\varphi = \pi/4$



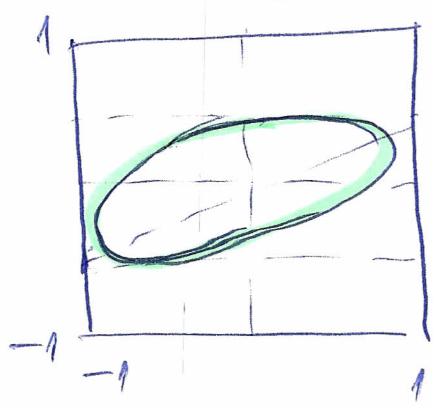
$\varphi = 3\pi/4$

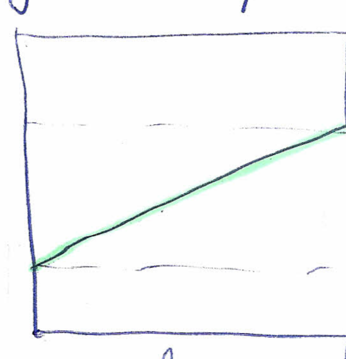


In case when the amplitude ratio changes, the slope changes for the rectangle inscribing the ellipse

ex: $A_1 = 1$
$A_2 = 0{,}5$
$f_x = f_y = 100\,Hz$
$\varphi = 45°$

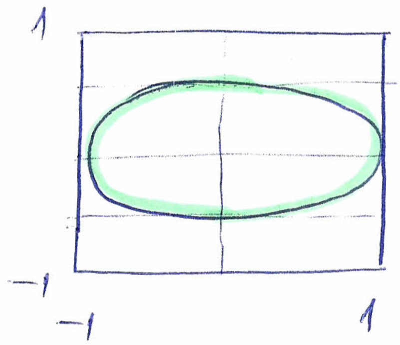
$\Rightarrow$



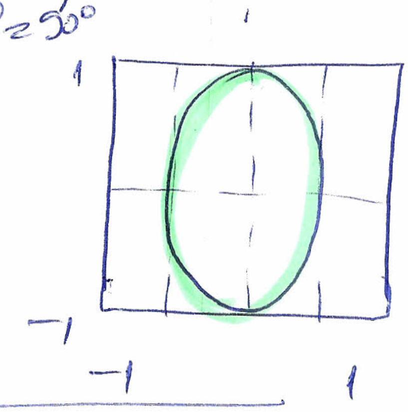


$\varphi = 0$

$A_1 = 1$
$A_2 = 0{,}5$
$f_x = f_y = 100\,Hz$
$\varphi = 90^\circ$



$A_1 = 0{,}5$
$A_2 = 1$
$f_x = f_y = 100\,Hz$
$\varphi = 90^\circ$



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(2) Lissajous figures $\omega_1 \neq \omega_2$ $(f_x \neq f_y)$

We play with the ratio:

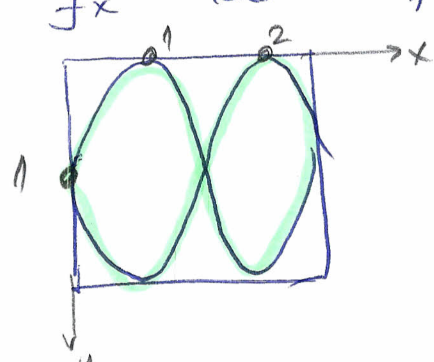
CH2 $\longrightarrow$ $\boxed{\dfrac{f_y}{f_x} = \dfrac{n_x}{n_y}}$
CH1 $\longrightarrow$

And also with ratios of amplitudes

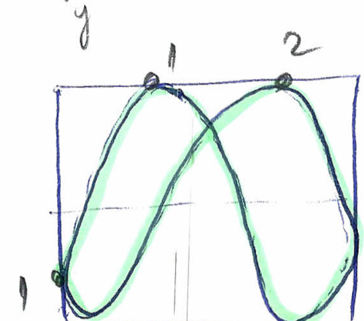
$$\frac{A_2}{A_1} = \frac{A_y}{A_x}$$

(a) $A_x = A_y = 0{,}5$
$f_1 = 100\,Hz$
$f_2 = 200\,Hz$
$\varphi = 0^\circ$

$$\frac{f_y}{f_x} = \frac{200}{100} = \frac{2}{1} = \frac{n_x}{n_y}$$



$\varphi = 45^\circ$

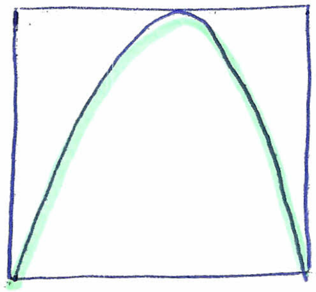


$\varphi = 90°$

$A_1 = A_2 = 0,5$

$f_1 = 100\ Hz$

$f_2 = 200\ Hz$

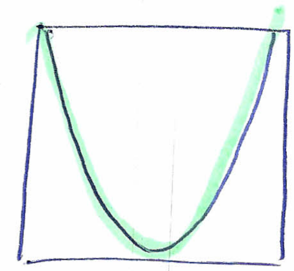


$\varphi = 270°$

$A_1 = A_2 = 0,5$

$f_1 = 100 Hz$

$f_2 = 200 Hz$

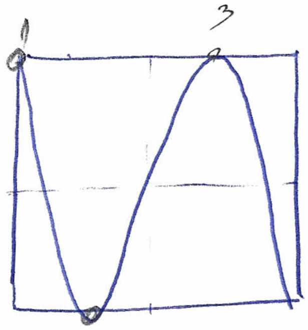


(b) $A_1 = A_2 = 0,5$

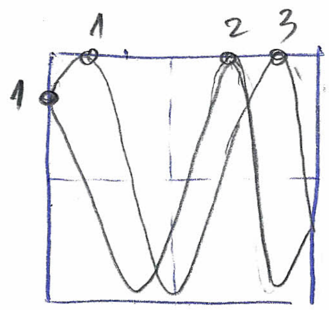
$f_1 = 100 Hz$

$f_2 = 300 Hz$

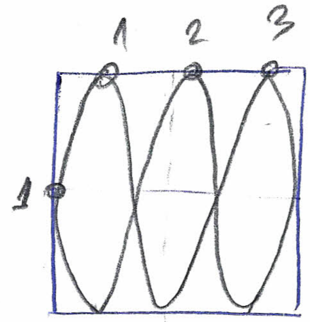
$$\Rightarrow \frac{f_y}{f_x} = \frac{3}{1} = \frac{n_x}{n_y}$$



$\varphi = 0$



$\varphi = 45°$



$\varphi = 90°$

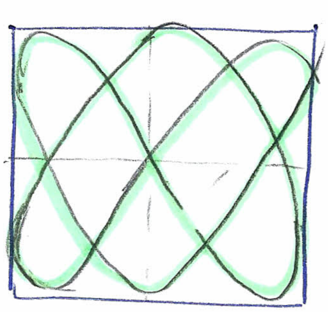
....

(c) $A_1 = A_2 = 0,5$

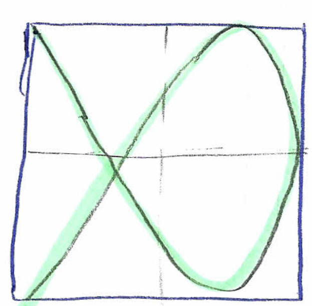
$f_1 = 200\ Hz$

$f_2 = 300 Hz$

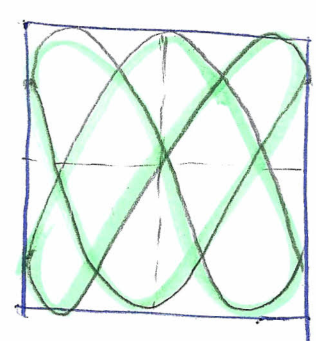
$$\frac{f_y}{f_x} = \frac{3}{2} = \frac{n_x}{n_y}$$



$\varphi = 0°$

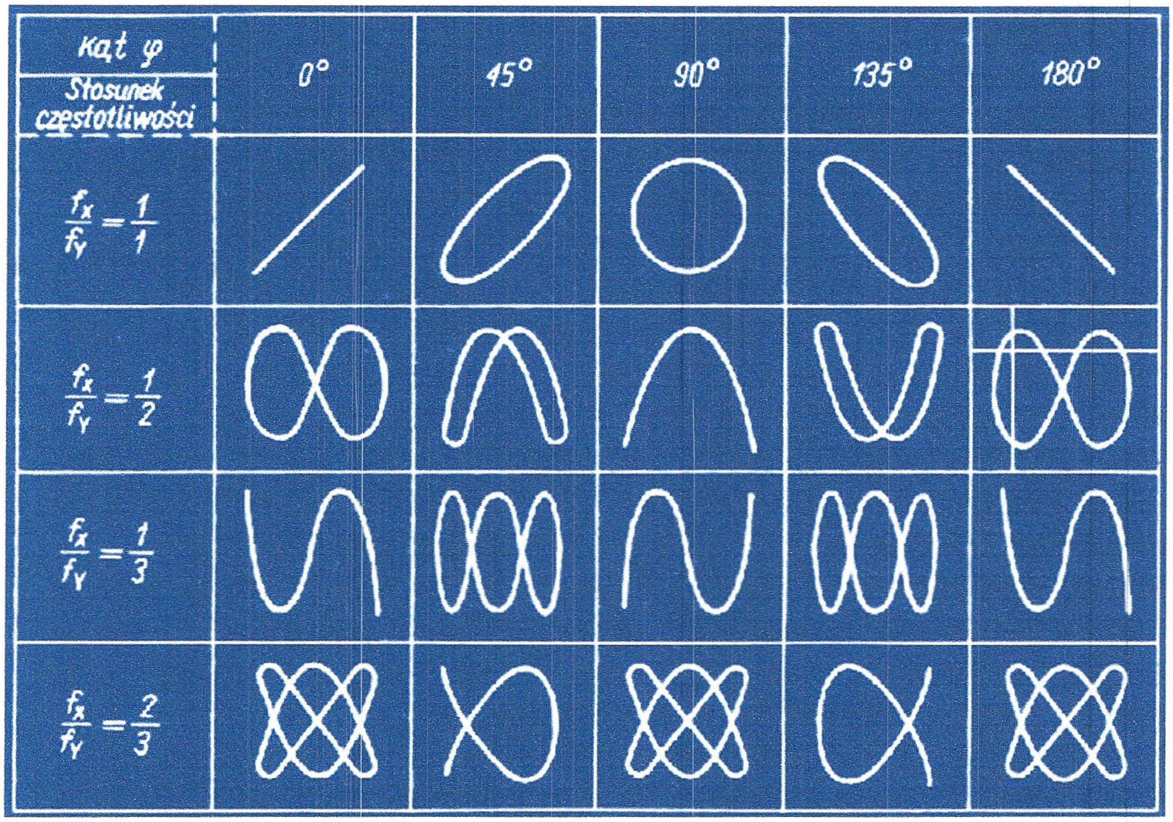


$\varphi = 45°$

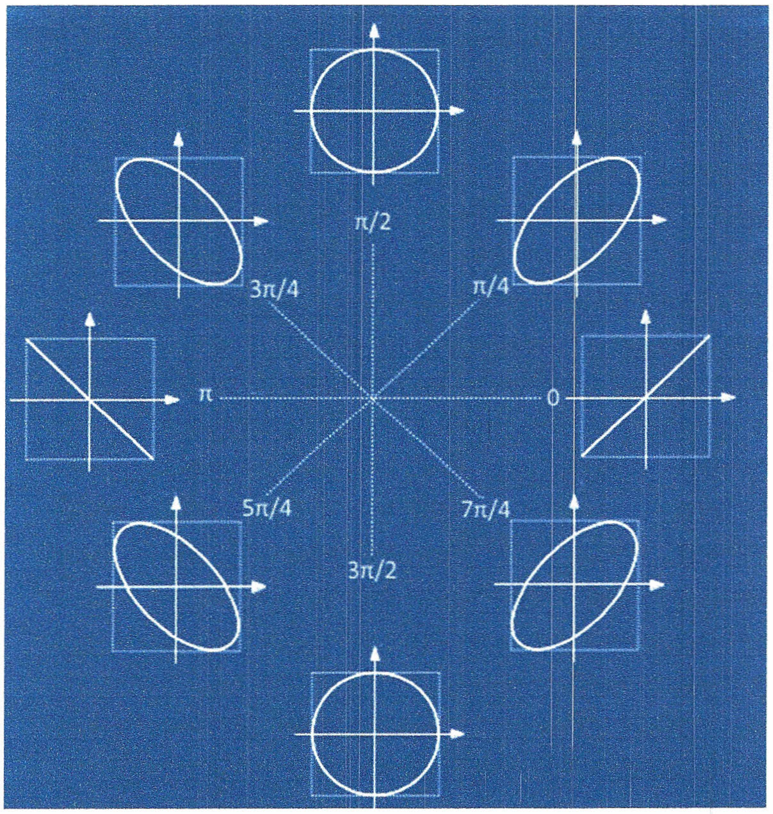


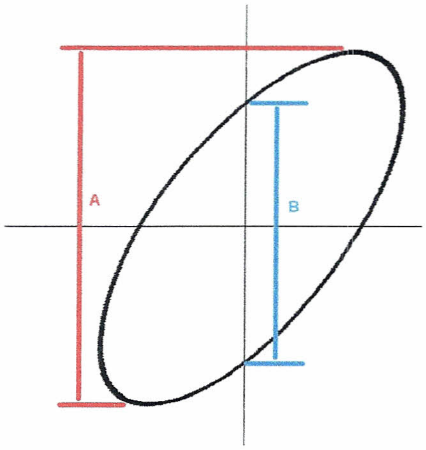
$\varphi = 90°$

Synthesis



Engineers can use the principles of Lissajous figures to precisely tune and set up the phase relation between a known reference signal and a signal to be tested.



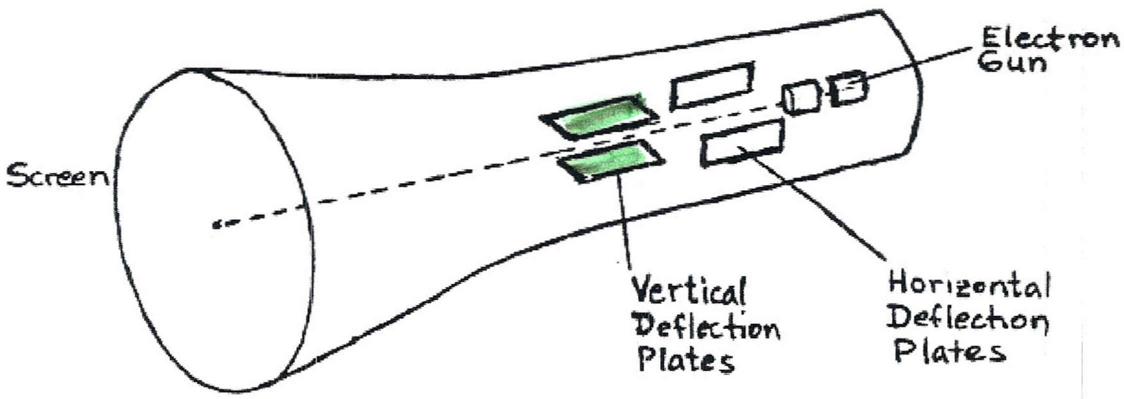


$$\sin \Phi = \frac{B}{A}$$

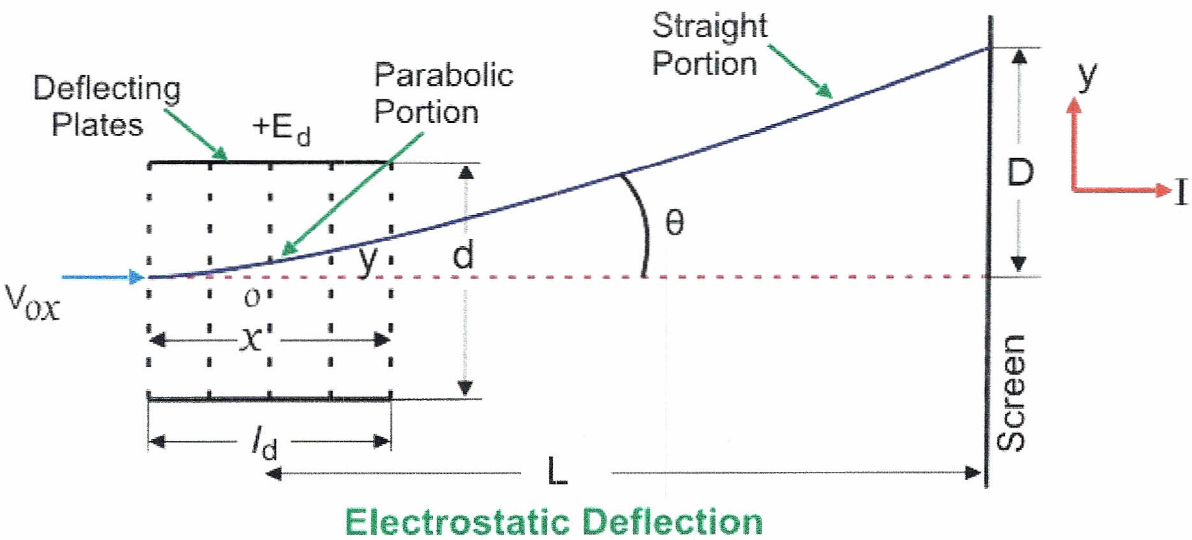
Φ = Phase Angle

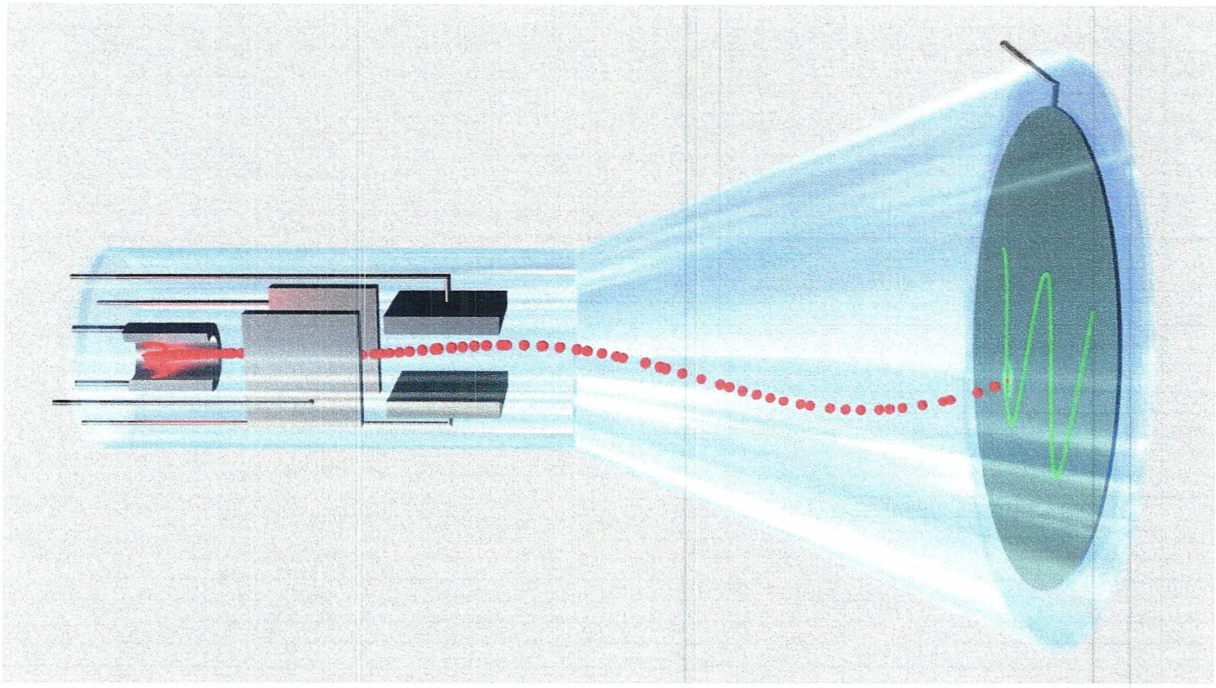
Electric setup:

The Oscilloscope:



Working principle: deflection of an electron in electric field $\vec{F} = q\vec{E}$





We will apply on x input: $X = U_x \sin(\omega_x t)$ and on Y input: $Y = U_y \sin(\omega_y t + \varphi)$ using two signal generators with variable frequencies and amplitudes.