CURRENT, RESISTANCE AND ELECTROMOTIVE FORCE

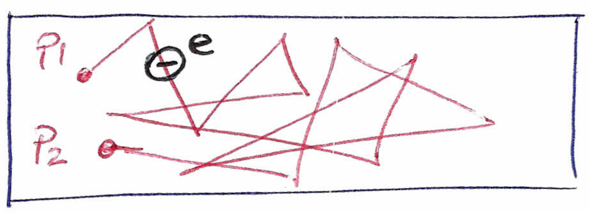
In the past chapters we studied the interaction of electric charges AT REST; the aim of this chapter is to study charges IN MOTION. An electric current consists in charges in motion from one region to another. If charges follow a conducting path that forms a closed loop, the path is called an ELECTRIC CIRCUIT.

1 CURRENT

A current is a motion of charge from one region to another following a path within a conducting material.

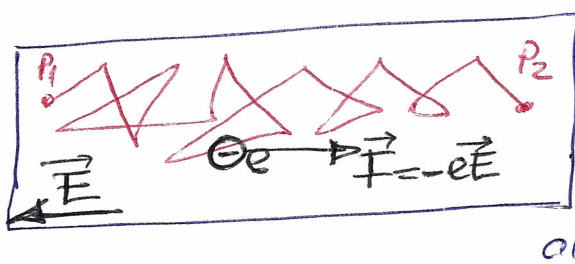
In electrostatic situation, the electric field $\vec{E}$ is zero everywhere within the conductor, there is no net current. However, in ordinary metals (Cu, Al, ..) the free electrons are randomly moving in all directions with speeds around 10^6 m/s. $\Rightarrow$ no net current.

$E=0$



conductor without external field E

$E \neq 0$



conductor within external field E, The motion remains almost random but a net displacement of charge is induced by $\vec{E}$ (net force $-e\vec{E}=\vec{F}_e$)

The almost random motion driven by $\vec{E}$ is described by a DRIFT VELOCITY $\vec{v}_d$ of the particles.

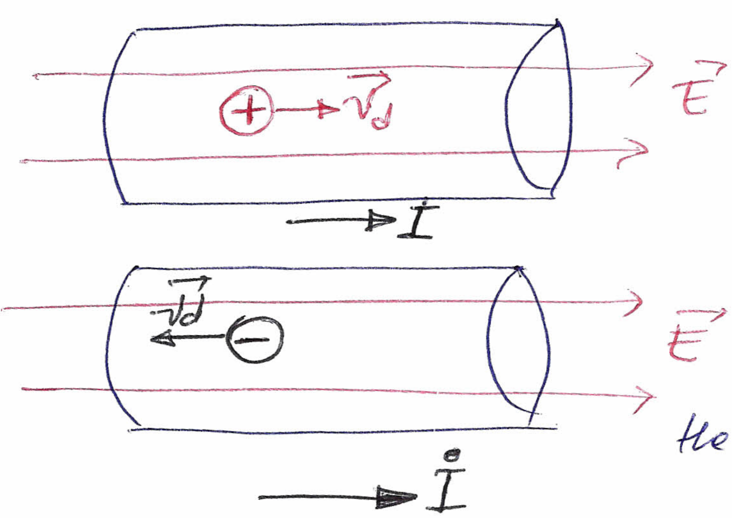
This drift velocity is much smaller than the free electron velocity $\boxed{v_d \simeq 10^{-4}\,m/s}$

The direction of the current flow

In different current-carrying materials the charges of moving particles can be positive or negative. In metals the charges are ELECTRONS and negative, while in ionized gas (plasma) or ionic solution one has both positive ions and negative electrons. In semiconductors (Si, Ge...) conduction is partially by electrons and partially by motion of VACANCIES (HOLES); these are sites of missing electrons and act as positive charges.

The charges are moving acted by the force $\vec{F} = q\vec{E}$ due to the electric field $\Rightarrow$

- positive charges are moving along $\vec{E}$ direction
- negative charges — " — oppositely to $\vec{E}$ direction



even e^- move against $\vec{E}$ direction, conventionally the current still point in the direction positive charges would flow.

Conventionally: the direction of the current flow is the one of positive charges would flow

We define the current through the cross-sectional area A to be the net charge flowing through the area per unit time

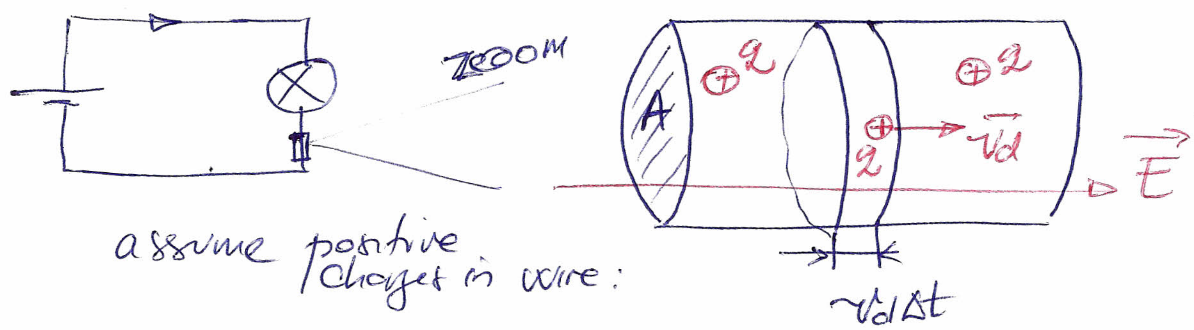
$$\boxed{I = \frac{dQ}{dt}} \qquad [I]_{SI} = \frac{C}{s} = A$$

(Ampère)

Multiples and submultiples

current in wires of a car start motor is $\sim 200A$
in radio, TV, electronics/gadgets $\sim mA, \mu A$
in computer circuits : nA, pA.

Current, drift velocity, Current density



assume positive charges in wire:

→ suppose n moving charges/unit volume (concentration of carriers)

→ assume that all particles move with same drift velocity v_d, and have same charge q

$$\Rightarrow \quad dQ = q(nAv_d\,dt) = nqv_dA\,dt$$

$$\boxed{I = \frac{dQ}{dt} = nqv_dA}$$

depends on the area of the conductor's cross-section

The current per unit cross-sectional area J is called the CURRENT DENSITY.

$$\boxed{J = \frac{I}{A} = nqv_d}$$

$$[J]_{SI} = \frac{A}{m^2}$$

If the moving charges are positive $\vec{v}_d$ points to $\vec{E}$ and if charges are negative $\vec{v}_d$ opposes to $\vec{E}$. And, the current I and the current density J still have same direction as $\vec{E}$ (don't depend on the sign of charge)

Therefore, we can replace q by $|q|$

$$\Rightarrow \boxed{I = \frac{dQ}{dt} = n|q|v_d A} \; ; \; \boxed{J = \frac{I}{A} = n|q|v_d}$$

We can define a vector current density $\vec{J}$ which includes the direction of the drift velocity

$$\boxed{\vec{J} = nq\vec{v_d}}$$

If $q > 0$ $\vec{J}$ points to $\vec{v_d}$
$q < 0$ $\vec{J}$ points against $\vec{v_d}$

but for $q > 0$ $\vec{v_d}$ points to $\vec{E}$
$q < 0$ $\vec{v_d}$ points against $\vec{E}$

$\Rightarrow$ in any situations $\vec{J}$ will point towards $\vec{E}$

For complex conductors with several kinds of moving charges $q_1, q_2, \ldots$ concentrations $n_1, n_2, \ldots$ and different drift velocities $v_{d1}, v_{d2}, \ldots$ (e.g. plasmas, ionic solutions) the net current and current density are obtain by addition

$$\boxed{\vec{J} = \sum_i n_i q_i \vec{v_{d_i}}}$$

2 RESISTIVITY

The current density $\vec{J}$ in a conductor depends on the electric field $\vec{E}$ and on the properties of the conductors material. This depence is quite complex, but for some materials, especially metals, at a given temperature, $\vec{J}$ is nearly directly proportional to $\vec{E}$. This relation has been discovered by the German physicist Ohm and the corresponding law has ~~his~~ name

$$\boxed{\vec{J} = \sigma\vec{E} = \frac{1}{\rho}\vec{E}}$$ Ohm's law

$$\boxed{\rho = \frac{E}{J}}$$ represents the <u>resistivity</u>

(describes how the material opposes to the current flow: higher ρ means larger E required for same J)

$$[\rho]_{SI} = \frac{\cancel{V}}{\cancel{m}} \cdot \frac{1}{\frac{A}{m^2}} = \frac{V}{A} \cdot m = \Omega \cdot m$$

$(\Omega = \text{Ohm})$

$$\boxed{\sigma = \frac{1}{\rho} = \frac{J}{E}}$$ represents the <u>conductivity</u>

$$[\sigma]_{SI} = (\Omega m)^{-1}$$

With respect to the resistivity, materials can be classified in
- CONDUCTORS (low ρ, high σ)
- SEMICONDUCTORS (intermediate ρ, σ)
- INSULATORS (low σ, high ρ)

<u>Resistivities at RT</u> (! $\rho = \rho(T)$; $\sigma = \sigma(T)$)

Conductors	$\rho(\Omega \cdot m)$
Metals	
Ag	$1.47 \cdot 10^{-8}$
Cu	$1.72 \cdot 10^{-8}$
Al	$2.75 \cdot 10^{-8}$
Steel	$20 \cdot 10^{-8}$
Hg	$95 \cdot 10^{-8}$
Alloys: Constantan (Cu 60% Ni 40%)	$49 \cdot 10^{-8}$
Semiconductors	
C (graphite)	$3.5 \cdot 10^{-5}$
Ge	0,60
Si	2300
Insulators	
glass	$10^{10} - 10^{14}$
wood	$10^{8} - 10^{11}$
quartz	$75 \cdot 10^{16}$

A material that obey Ohm's law is called Ohmic conductor or linear conductor. For such material, at a certain T, ρ is constant that not depends on E.

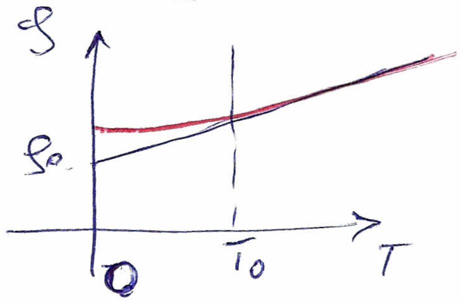
Materials which do not obey linear $J(E)$ behaviour are non-ohmic or non-linear (i.e. diodes, tunnel junctions, ..)

Resistivity and temperature

The temperature variation of resistivity depends on the type of conducting materials:

① In metals ρ increases when T increases, linearly

$$\boxed{\rho(T) = \rho_0 [1 + \alpha (T - T_0)]}$$

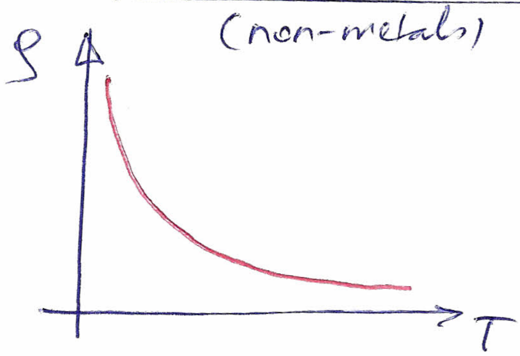


ρ_0 – reference resistivity at a refference temperature T_0

α = temperature coefficient of resistivity $[\alpha]_{SI} = (^\circ C)^{-1}$

Material	$\alpha\,[(^\circ C)^{-1}]$
Al	0.0039
Cu	0.00393
Ag	0,0038

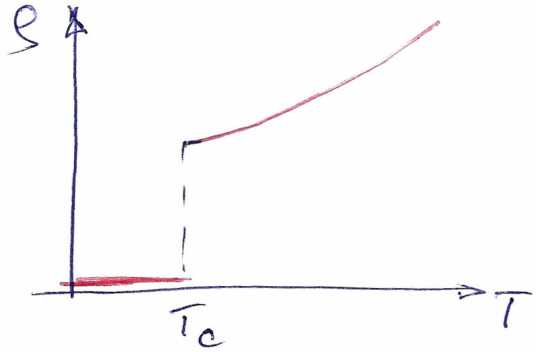
② $\boxed{\text{Semiconductors}}$ (non-metals) ρ decreases when T increases



Increasing T in semiconductor, more carriers became available for conduction $\Rightarrow$ resistivity decreases.

Measuring resistivity of a small semiconductor crystal is a sensitive measure of temperature; this is a principle of a thermometer called THERMISTOR

(3) Superconductors



Special Class of materials which below a critical temperature loss their resistivity $\Rightarrow$ infinite conductivity

(Kamerlingh Ones) discovered that Hg below 4,2K becomes supraconductor

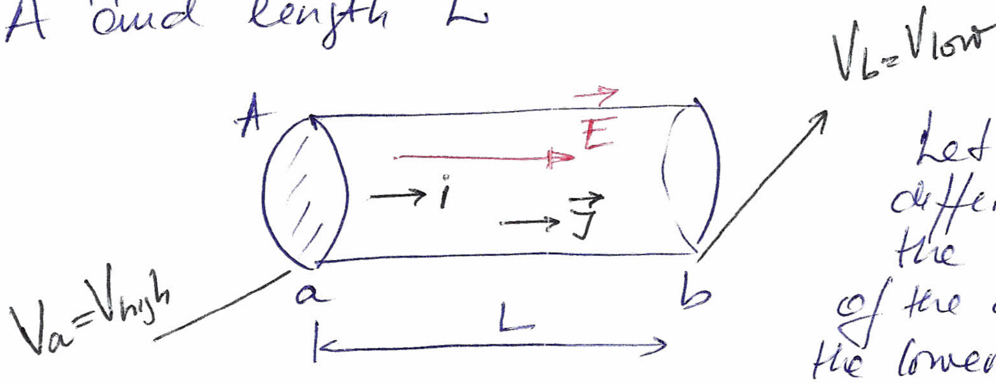
metallic supraconductors (low Tc)
oxidic complex supraconductors (high Tc)
$YBa_2Cu_3O_4$ (>1987) $T_c = 77K$
current record $T_c = 138\,K$

– see later in a next chapter more about supraconductivity

[3] Resistance

From the Ohm's law $\vec{J} = \sigma \vec{E} \iff$
$$\vec{E} = \rho \vec{J}$$

Suppose that our conductor is a wire with section A and length L



Let V = the difference between the higher potential of the conductor and the lower potential of the conductor, so V is positive

The direction of the current is from higher-potential to lower potential ends.

$$V_{ab} = V_a - V_b = E \cdot L \Rightarrow E = \frac{V}{L}$$

From Ohm's law $\Rightarrow$

$$\frac{V}{L} = \rho \cdot \frac{\dot{I}}{A} \quad \text{or} \quad V = \rho \frac{L}{A} \dot{I}$$

The ratio $\boxed{\frac{V}{I} = R}$ for a particular conductor is called <u>RESISTANCE</u> $\Rightarrow$ $\boxed{R = \rho \frac{L}{A}}$

relationship between resistance and resistivity.

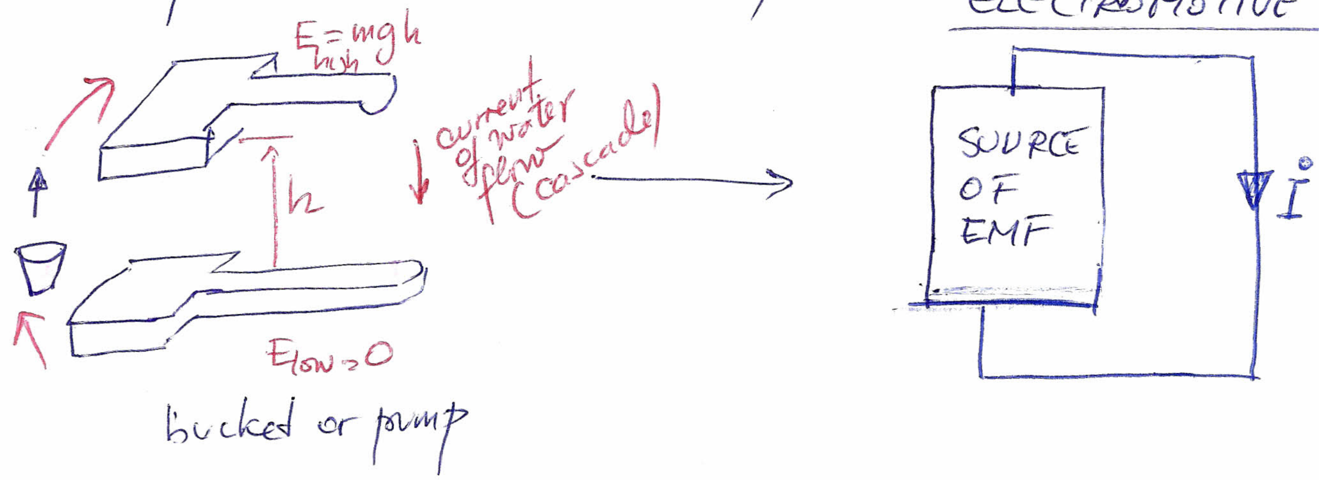
often called macroscopical form of Ohm's law

<u>Obs</u>: The equation $\boxed{R = \frac{V}{I}}$ defines the resistance for any conductor, whether or not it obeys Ohm's law, but only when R is constant we can call this relationship Ohm's law.

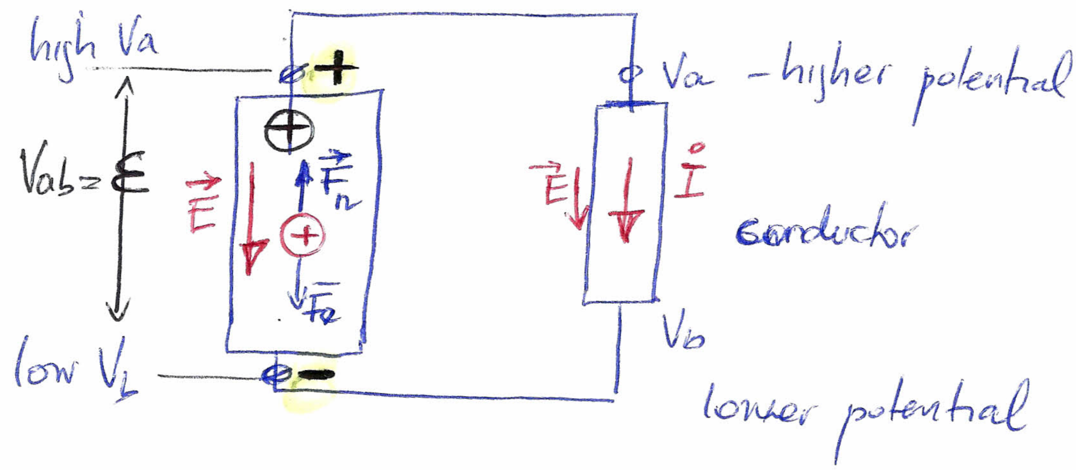
4 ELECTROMOTIVE FORCE AND CIRCUITS

For a conductor to have a steady current it must be a part of a path that forms a <u>COMPLETE CIRCUIT</u>

This circuit requires a devices somewhere in the loop that acts as a water pump in a water fountain, and provides to carriers the energy necessary to "uphill" from lower to higher potential energy $\Rightarrow$ <u>SOURCE OF ELECTROMOTIVE FORCE (EMF)</u>



IDEAL EMF source



$\vec{F}_n - \vec{F}_e$ does work on charges

$$\boxed{\mathcal{E} = V_{ab} = IR}$$

The electric field inside of the source is $E \propto V_a - V_b$ which wold act with a force $\vec{F}_e$ on the direction of $\vec{E}$ (electrostatic force). However, an additional force is required ($\vec{F}_n$ = non-electrostatic force) to "uphill" the electrons to higher potential V_a from lower V_b and to maintain the potential difference between terminals a and b.

The origin of $\vec{F}_n$ depends on the type of the source. In a generator, it result from magnetic field forces of moving charges. In battery or fuel-cell it is associated to diffusion processes and varying electrolyte concentrations resulting from chemical reactions.

Internal rezistance

Above we descrited an idealized case. In reality, carriers moving inside an EMF source encounters rezistance $\Rightarrow$ internal rezistance of the source r

$$\Rightarrow \boxed{V_{ab} = \mathcal{E} - Ir}$$

terminal voltage

potential drop across the internal rezistance

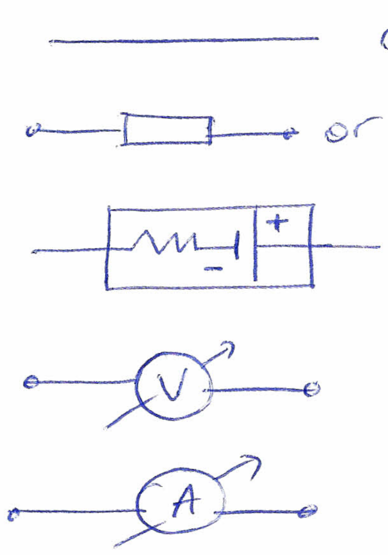
$\Rightarrow$ the current in the external circuit with a EMF source having an internal resistance r

$$V_{ab} = iR \quad \Rightarrow \quad \mathcal{E} - Ir = IR \Rightarrow$$

$$\boxed{I = \frac{\mathcal{E}}{R+r}}$$

Obs: The battery is not a current source. The larger is R, the lower is I for same $\mathcal{E}$.

Symbols for circuit diagrams



conductor with $R=0$

or

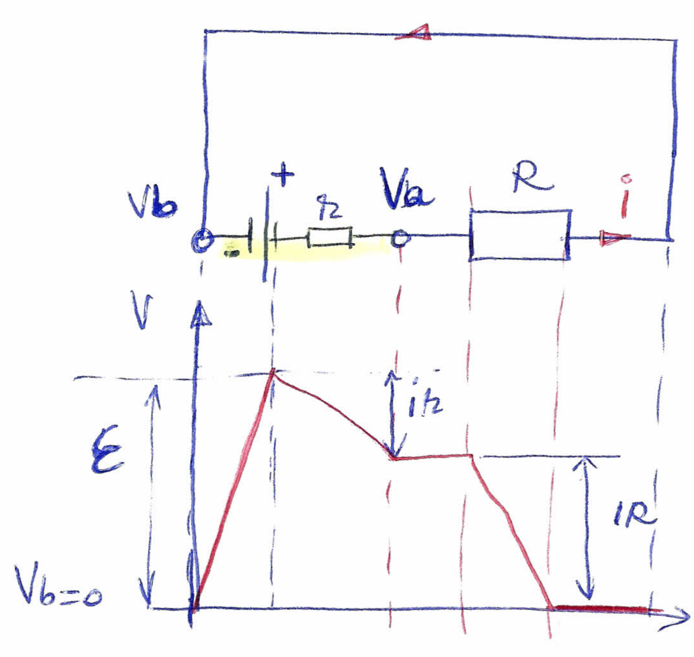
resistor ($R \neq 0$)

source of EMF with internal resistance r

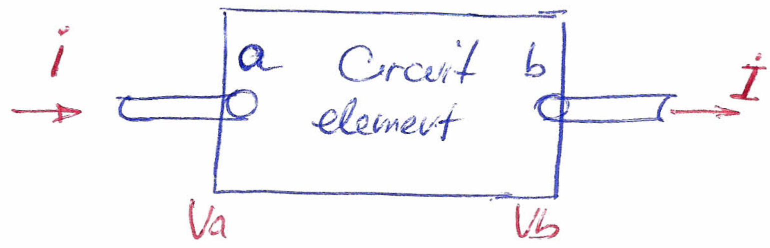
voltmeter (measures the potential difference between terminals)

ampermeter (measures current through it)

Potential changes across a circuit



5 ENERGY AND POWER IN ELECTRIC CIRCUITS



$$P = \frac{d}{dt}\Big(\underbrace{dQ\, V_{ab}}_{}\Big)$$

potential energy change by V_{ab} on elementary charge dQ.

<u>Power</u>

$$\boxed{P = V_{ab}\, I}$$

rate at which energy is delivered or extracted from a circuit element

$$[P] = V \cdot A = 1 \text{ Watt}$$

pure resistance $\quad V_{ab} = R\, i$

$$\Rightarrow \boxed{P = \frac{V_{ab}^2}{R} = R\, I^2}$$

power delivered to a resistor

what becomes this energy?

The moving charges collides with atoms in resistor and some of energy is converted in internal energy of material $\Rightarrow$ heat, temperature increase (heat dissipation)

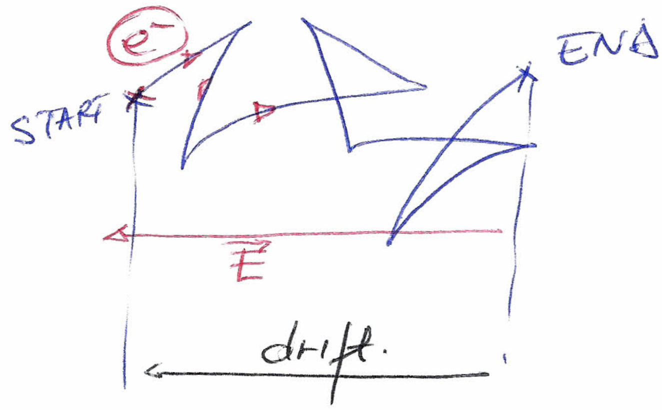
$\Rightarrow$ <u>Joule effect</u>

6 Theory of metallic conduction

What is the microscopic origin of conductivity?

We use a simple model that treats the electrons as simple classical particles ignoring quantum-mechanics effects. Using this model, we derive an expression of the metal resistivity.

Model: - each atom in the metal crystal gives up one ore more free electrons
- the free electrons collide with stationary ions from time to time.
- if there is no electric field, the velocities or moving electrons are random, and the average displacement is zero.
- under electric field a net displacement appears



with $\vec{E}$, random motion + drift

$v_e = 10^6\ m/s$

$v_d \sim 10^{-4}\ m/s$

The average time between two collisions is called MEAN FREE TIME, τ. The distance travelled in this time is called MEAN-FREE-PATH

$$\rho = \frac{E}{J} \quad \text{by deffinition and}$$

$$\vec{J} = nq\vec{v_d} \qquad \text{where } q = -e$$

We need to relate $\vec{v_d}$ to $\vec{E}$. The value of $\vec{v_d}$ is determined by a steady-stable condition in which, in overage, the velocity changes due to the force of $\vec{E}$ field are just balanced by the velocity losses by collisions.

Consider at $t=0$ no field $\vec{E}=0$ $\Rightarrow$

$$\vec{v}_0 = 0$$

Turning on the field the electron feels a force:

$\vec{F} = q\vec{E}$ which gives an acceleration

$$\vec{a} = \frac{\vec{F}}{m} = q\frac{\vec{E}}{m}$$

In the time τ between two collisions the electron aquires a velocity

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

If we average $\Rightarrow$

$$\underbrace{\langle\vec{v}\rangle_{av}}_{\vec{v}_d} = \underbrace{\langle\vec{v}_0\rangle_{av}}_{0} + a\tau \qquad (\tau = \langle t\rangle)$$

$\vec{v}_d \rightarrow$ the speed the electron maintains after collisions.

$$\Rightarrow \vec{v}_d = \vec{a}\tau = \frac{qE\tau}{m} \Rightarrow$$

$$\boxed{\vec{J} = nq\vec{v}_d = \frac{nq^2\tau}{m}\vec{E}} = \frac{1}{\rho}\vec{E}$$

$q = -e$

$$\Rightarrow \boxed{\rho = \frac{m}{ne^2\tau}}$$

<u>Temperature dependence of ρ</u>

In an ideal crystal with no atoms out of place the correct quantum mechanical analysis would let the electrons to move trough crystal without collisions at all. But, the atoms at a finite temperature vibrate, and the

vibration amplitude increase with increasing T
$\Rightarrow$ collisions become more frequent and the mean free time τ decreases. $\Rightarrow \rho \nearrow$

- In a supraconductor, roughly speaking, there are no collisions $\Rightarrow \rho \to 0$
- In a pure semiconductor, n is not constant but rapidly increasing with temperature $T \Rightarrow \rho \nearrow$. At low temperature, n is very small, so $\rho \to 0$, the material becoming insulator.

Obs: Electrons gain energy between collisions due to the work done on them by the electric field $\vec{E}$. During collisions they transfer some of this energy to atoms of the material of conductor. This leads to an increase of the material's internal energy and temperature, this is why wires carrying current get warm. If the electric field in the material is large enough, an electron between two collisions can gain enough energy to knock off other electrons normally bonded to atoms in material. Thes can knock more electrons and an avalanche of current is determined. This is the basis of the dielectric breakdown of insulators.

DIRECT-CURENT CIRCUITS

Inside of electronic devices (TV, computers, gadgets...) we find circuits that are more complex, as interconected networks.

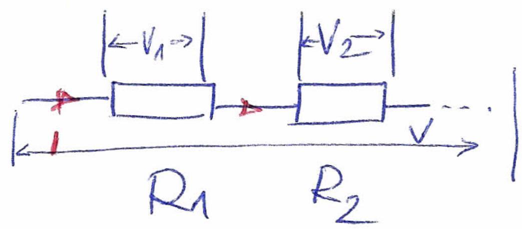
There are two classes of circuits:

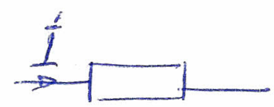
DC circuits (dirrect-current) in which the direction of the current does not change in time.

AC circuits (alternating current) circuits where the current oscillates back and forth.

1 Resistors in series and parallel

Series





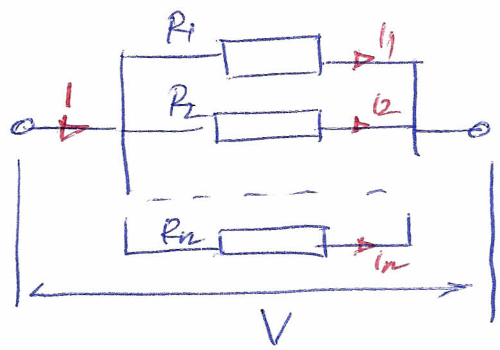
$$\boxed{R_e = \sum_i R_i}$$

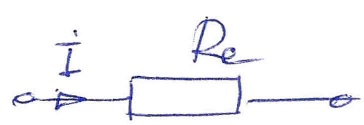
Same I, voltage drop distributed

$$V = V_1 + V_2 + \dots = \sum V_i$$

$$R_e = \frac{V}{I} = \frac{\sum V_i}{I} = \sum R_i$$

parallel





Same V, I distributed $\quad I = \sum_i I_i$

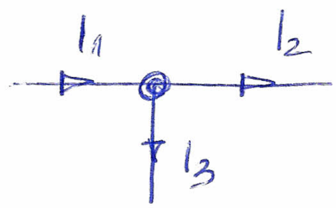
$$\Rightarrow \sum \frac{V}{R_i} = V \sum \frac{1}{R_i} = \frac{V}{R_e} \Rightarrow$$

$$\boxed{\frac{1}{R_e} = \sum_i \frac{1}{R_i}}$$

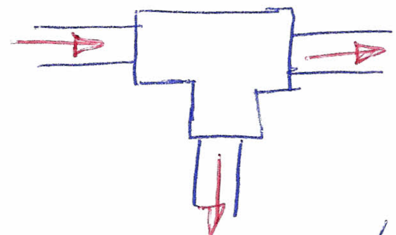
[2] Kirchhoff's rules

① **Kirchhoff junction rule**: The algebraic sum of all the currents into any junction is zero

$$\boxed{\sum I = 0}$$



$\Longleftrightarrow$ water pipe analogy



flow in = flow-out

This is based on the Conservation of the electric charge

Convention: entering (+)
out (−)

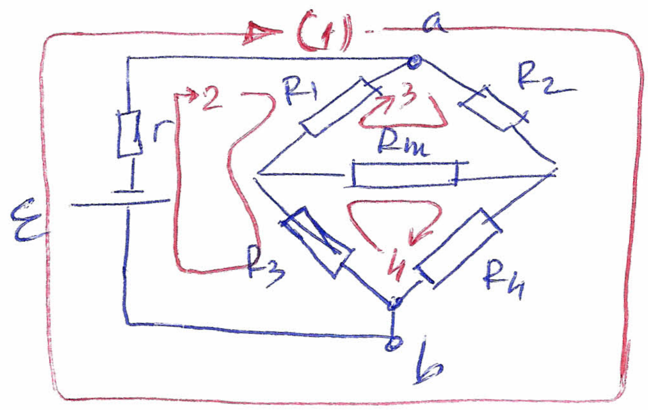
$$I_1 - I_2 - I_3 = 0$$

② **Kirchhoff loop rule**

The algebraic sum of all the potential differences in any loop, including those with EMF and those of resistive elements must be zero.

$$\boxed{\sum V = 0}$$

in closed loop



Sign convention

for a resistor: when we travel in the direction of the current IR is negative because the current goes in the direction of decreasing potential.

For EMF source:

When travelling from (−) to (+) emf is positive and from (−) to (+) negative

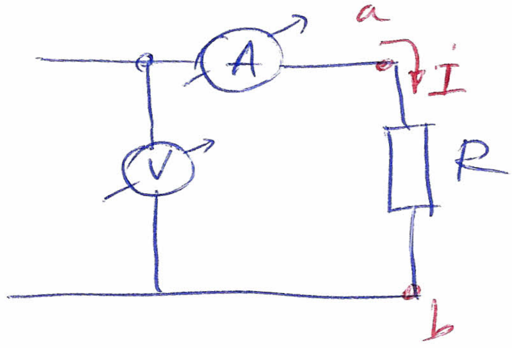
The loop rule is a statement that the electrostatic force is conservative. Suppose we go around a loop measuring potential differences across successive circuit elements as we go. When we return to the starting point we must find that the algebraic sum of all these differences is zero.

3 Electrically measuring instruments

AMMETERS (A) → measure I which passes through it ⇒ must be inserted in series in a circuit

volt-metters (V) → measure potential difference or voltage.
- it is connected in-parallel
- an ideal voltmeter has an internal resistance ∞

Combined A-V circuits

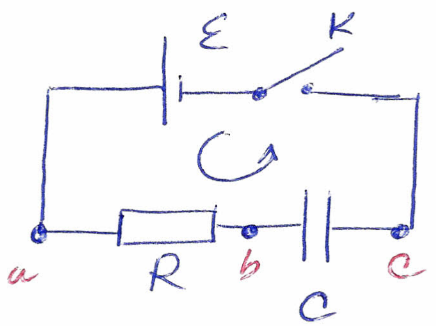


A − measures i

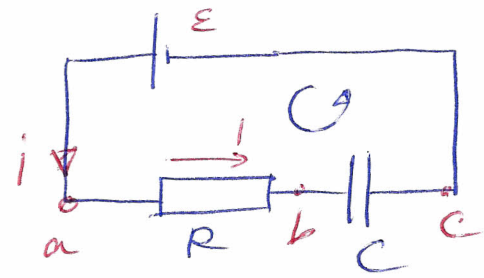
V − measures V_{ab}.

4 R-C circuits

Charging a capacitor



$t=0$ K opened
$t=0_+$ K closed



Kirchhoff loop rule

$$\varepsilon = V_R + V_C$$ at $t=0$ $V_C = 0 \Rightarrow$

$$\varepsilon = I_{0_+} R$$

$t > 0_+$ V_C increases so V_R has to decrease

$$V_R = V_{ab} = iR$$

$$V_{bc} = V_C = \frac{q}{C}$$

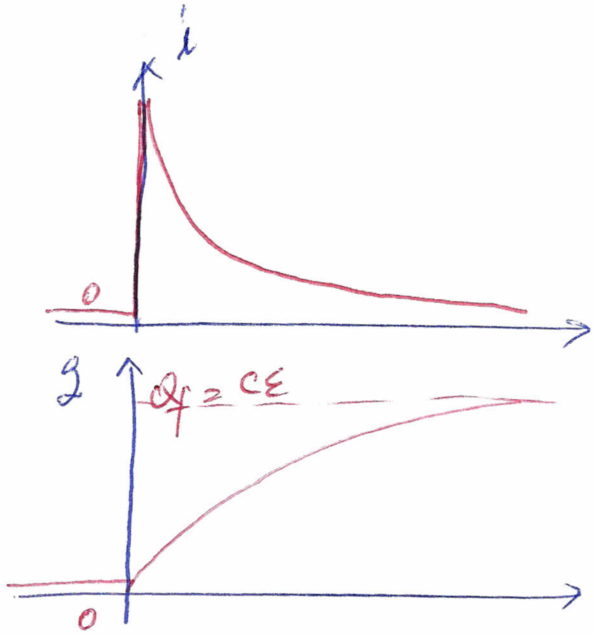
$$\Rightarrow \varepsilon - iR - \frac{q}{C} = 0 \qquad i = \frac{dq}{dt} \Rightarrow$$

$$\boxed{\frac{dq}{dt} - q/(RC) - \frac{\varepsilon}{R} = 0}$$ differential 1st order eq.

$$\left(\frac{dq}{dt}\right)_{t=0} = \frac{\varepsilon}{R}$$

solution: $$\boxed{q = C\varepsilon\left(1 - e^{-t/RC}\right)}$$

$$\boxed{i = \frac{dq}{dt} = \frac{\varepsilon}{R} e^{-t/RC} = I_0 e^{-t/RC}}$$



Time constant:

$$\boxed{\tau = RC}$$

the time after which the current decreased e times

Appendix Solving the differential eq:

$$\boxed{\frac{dq}{dt} - \frac{1}{RC} q - \frac{\mathcal{E}}{R} = 0}$$

we rearange as: $\frac{dq}{q - C\mathcal{E}} = -\frac{dt}{RC}$ and integrate both sides

$$\int_0^q \frac{dq}{q - C\mathcal{E}} = -\int_0^t \frac{dt}{RC} \Rightarrow$$

$$\ln\left(\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right) = -\frac{t}{RC}$$ exponentiating both sides

$$\Rightarrow \frac{q - C\mathcal{E}}{-C\mathcal{E}} = e^{-t/RC} \Rightarrow$$

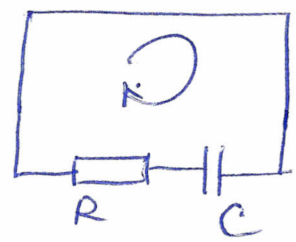
$$q = C\mathcal{E}\left(1 - e^{-t/RC}\right) = Q_f\left(1 - e^{-t/RC}\right)$$

$$i = \frac{dq}{dt} = \frac{\mathcal{E}}{R} e^{-t/RC} = I_0 e^{-t/RC} = I_0 e^{-t/\tau}$$

$$\tau = RC$$

Discharging a capacitor

Now, we suppose that after the capacitor aquired charge Q_0 we close the circuit



The capacitor discharges through the resistor

$$iR + \frac{q}{C} = 0$$

$V_R = iR$

$V_C = \frac{q}{C}$

$$\frac{dq}{dt} R + \frac{q}{C} = 0 \iff$$

$$\frac{dq}{dt} + \frac{q}{RC} = 0$$

$$\int_{Q_0}^{q} \frac{dq}{q} = -\int_0^t \frac{dt}{RC} \Rightarrow \ln \frac{q}{Q_0} = -\frac{t}{RC} \Rightarrow$$

$$\boxed{q = Q_0 e^{-t/RC}}$$

$$i = \frac{dq}{dt} = -\frac{Q_0}{RC} e^{-t/RC} = I_0 e^{-t/RC}$$

$$\boxed{i = I_0 e^{-t/RC}}$$